Lecture 2

Prime number estimates of Chebyshev

Recall that

$$\pi(x) := \#\{p \leq x : p \text{ prime}\} = \sum_{p \leq x} 1$$

We will also consider the following weighted sums

$$\theta(x) := \sum_{p<x} \log p$$

$$\psi(x) := \sum_{n \leq x} \Lambda(n)$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{o/w} \end{cases}$$

denotes the <u>von Mangoldt function</u>.

(Note that this is well-defined by the fundamental theorem of arithmetic.)

As we shall see, these three summatory functions are closely related. But first we state the following main result:

<u>Theorem 1</u> (Chebyshev) If $x \geq 2$, then $\psi(x) \asymp x$.

The proof we give below establishes only that there is an x_0 such that $\psi(x) \asymp x$ uniformly for $x \geq x_0$. However, both $\psi(x)$ & x are bounded away from 0 & ∞ on $[2, x_0]$, and hence the implicit constants can be adjusted so that $\psi(x) \asymp x$ uniformly for $x \geq 2$.

Corollary 1: For $x \geq 2$,

$$\theta(x) = \psi(x) + O(x^{1/2}) \quad \& \quad \pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

Corollary 2: For $x \geq 2$,

$$\theta(x) \asymp x \qquad \& \qquad \pi(x) \asymp \frac{x}{\log x}$$

Before proving any of these results we recall

Proposition (Summation by Parts)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Set

$$A(t) = \sum_{n \leq t} a_n \qquad (t > 0).$$

If $f(t)$ is a continuously differentiable function on $[1, x]$, then

$$\sum_{y < n \leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t) f'(t)\,dt$$

and in particular $\displaystyle\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t) f'(t)\,dt$.

Exercise 1: Prove this proposition. Hint: Write

$$\sum_{y < n \leq x} a_n f(n) = \underbrace{\int_{y^-}^{x} f(t)\,dA(t)}_{\text{Riemann-Stieltjes integral.}} \text{ and integrate by parts.}$$

We now show that Corollary 1 follows easily from Theorem 1.

(Corollary 2 is of course an immediate consequence of Corollary 1).

Proof of Corollary 1

Clearly

$$\psi(x) = \sum_{p^k \le x} \log p = \sum_{k=1}^{\infty} \sum_{p \le x^{1/k}} \log p = \sum_{k=1}^{\infty} \theta(x^{1/k})$$

But $\theta(y) \le \psi(y) \ll y$ (by Theorem 1)

Note: $\theta(x^{1/k}) = 0$ if $x^{1/k} < 2$ $\Updownarrow$ $k > \log x / \log 2$.

Hence

$$\psi(x) - \theta(x) = \sum_{k=2}^{\infty} \theta(x^{1/k}) \ll x^{1/2} + x^{1/3} \log x \ll x^{1/2}$$

As for $\pi(x)$:

Partial Summation

$$\pi(x) = \sum_{n \le x} a_n f(n) = \theta(x) \frac{1}{\log x} + \int_2^x \theta(t) \frac{1}{t(\log t)^2} \, dt$$

where $a_n = \begin{cases} \log p & \text{if } n = p \\ 0 & \text{o/w} \end{cases}$ & $f(t) = \frac{1}{\log t}$.

$$= \frac{\psi(x)}{\log x} + O\left(\frac{x^{1/2}}{\log x}\right) + \int_2^x \theta(t) \frac{1}{t(\log t)^2} \, dt$$

Result follows if we can show that

$$\int_2^x \theta(t) \frac{1}{t(\log t)^2} \, dt \underset{\uparrow}{\ll} \int_2^x \frac{dt}{(\log t)^2} \underset{?}{\ll} \frac{x}{(\log x)^2}$$

Immediate.

Exercise 2 Verify by writing $\int_2^x = \int_2^{x^{1/2}} + \int_{x^{1/2}}^x$.

Proof of Theorem 1

First an important lemma.

<u>Lemma 1</u>: For every $n \in \mathbb{N}$, $\displaystyle\sum_{d|n} \Lambda(d) = \log n$.

<u>Proof</u>: Write $n = \prod_{p|n} p^{\ell_p}$.

$$\sum_{d|n} \Lambda(d) = \sum_{p^k|n} \log p = \sum_{p|n} \sum_{k=1}^{\ell_p} \log p = \sum_{p|n} \log p^{\ell_p} = \log \prod_{p|n} p^{\ell_p} = \log n \quad \square$$

Let $T(x) := \sum_{n \le x} \log n$. By the integral test we see that

$$\int_1^N \log t \, dt \le T(N) \le \int_1^{N+1} \log t \, dt$$

for any $N \in \mathbb{N}$. Since $\int \log x \, dx = x \log x - x$, it follows easily that

$$T(x) = x \log x - x + O(\log^2 x) \qquad (*)$$

for $x \ge 1$. The link between $T(x)$ & $\psi(x)$ is given by

<u>Lemma 2</u>: For every $x > 0$, $\displaystyle T(x) = \sum_{n \le x} \psi(x/n)$.

<u>Proof</u>: Observe that

$$\sum_{n \le x} \psi(x/n) = \sum_{n \le x} \sum_{m \le x/n} \Lambda(m) = \sum_{nm \le x} \Lambda(m) = \sum_{N \le x} \sum_{m|N} \Lambda(m) \overset{\text{Lemma 1}}{=} \sum_{N \le x} \log N = T(x)$$

$\square$

We are now ready to prove Theorem 1.

Proof (of Theorem 1)

Suppose $x \geq 4$. By (*) we see that

$$T(x) - 2T(x/2) = x\log x - x + O(\log x) - 2\left(\frac{x}{2}\log\frac{x}{2} - \frac{x}{2} + O\left(\log\frac{x}{2}\right)\right)$$

$$= x\log 2 + O(\log x).$$

On the other hand, Lemma 2 gives

Note: If $n > \frac{x}{2}$ $\Rightarrow \psi\left(\frac{x}{2n}\right) = 0$.

$$T(x) - 2T(x/2) = \sum_{n \leq x} \psi(x/n) - \sum_{n \leq x} 2\psi(x/2n)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\psi(x/n) = \psi(x) - \psi(x/2) + \cdots$$

Since ψ is an increasing function, this is an alternating series of decreasing terms. It follow that for any <u>even</u> k:

$$T(x) - 2T(x/2) \geq \psi(x) - \psi(x/2) + \cdots + \psi(x/_{k-1}) - \psi(x/k),$$

while for any <u>odd</u> k:

$$(***) \quad T(x) - 2T(x/2) \leq \psi(x) - \psi(x/2) + \cdots - \psi(x/_{k-1}) + \psi(x/k).$$

Taking $k=1$ above gives:

$$\psi(x) \geq T(x) - 2T(x/2) = x\log 2 + O(\log x)$$

Getting an upper bound on $\psi(x)$ is trickier! Notice first that taking $k=2$ above gives:

$$\psi(x) - \psi(x/2) \leq T(x) - 2T(x/2) = x\log 2 + O(\log x).$$

It follow that for any $j \in \mathbb{N}$;

$$\psi(x/2^{j-1}) - \psi(x/2^{j}) \leq \frac{x}{2^{j-1}}\log 2 + O\left(\log \frac{x}{2^{j-1}}\right) = \frac{x}{2^{j-1}}\log 2 + O(\log$$

If we now choose k such that $\frac{x}{2^k} < 4 \leq \frac{x}{2^{k-1}}$ and note that $k \ll \log x$

it follows that

$$\psi(x) - \psi(x/2^k) = \sum_{j=1}^{k} \psi(x/2^{j-1}) - \psi(x/2^{j})$$

Since $\psi(\frac{x}{2^k}) \leq \psi(4)$.

$$\leq x\log 2\left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{k-1}}\right) + O((\log x)^2)$$

$$\Rightarrow \quad \psi(x) \leq 2x\log 2 + O((\log x)^2). \quad (****)$$

$\square$

<u>Bertrand's Postulate</u> (for sufficiently large x).

Taking $k=3$ in $(***)$ gives

$$\psi(x) - \psi(x/2) + \psi(x/3) \geq T(x) - 2T(x/2)$$

$$= x\log 2 + O(\log x).$$

Since from $(****)$ we know

$$\psi(x/3) \leq \frac{2}{3}x\log 2 + O((\log x)^2)$$

it follows that

$$\psi(x) - \psi(x/2) \geq \frac{1}{3}x\log 2 + O((\log x)^2)$$

Corollary 1

$$\Rightarrow \quad \theta(x) - \theta(x/2) \geq x\frac{\log 2}{3} + O(x^{1/2}) \quad (\text{as } x \to \infty).$$

It follows that for all sufficiently large x

$$\theta(2x) - \theta(x) = \sum_{x < p \leq 2x} \log p > 0.$$

In particular, there must exist a prime p in $(x, 2x]$.

$\square$