Theorem 2

If E is a Lebesgue measurable subset of $\mathbb{R}$, with $m(E)>0$, then its difference set:

$$E-E=\{d : d=x-y \text{ with } x,y\in E\}$$

contains an open interval centered at the origin.

We first consider the following finite (quantitative) version:

Lemma If $E\subseteq I$, I open interval and $m(E)>\frac{3}{4}m(I)$, then $m(E\cap(E+d))>0$ for all $|d|<\frac{1}{2}m(I)$.

[Note: $d\in E-E \Leftrightarrow E\cap(E+d)\neq\emptyset$]

Proof of Lemma: Exercise! [Homework 1]

$$m(E\cap(E+d)) = m(E)+m(E+d)-m(E\cup(E+d))$$

$$\geq 2m(E)-\left(m(I)+|d|\right)$$

$E\cup(E+d)\subseteq I\cup(I+d)$

$$\geq \frac{1}{2}m(I)-|d|$$

$$>0 \quad \text{if } |d|<\frac{1}{2}m(I) \qquad \square$$

Proof of Theorem 2:

By the Lemma, it suffices to show that there exists an open interval I such that $m(E \cap I) > \frac{3}{4} m(I)$.

Let $\varepsilon > 0$, $\exists$ open G with $E \subseteq G$ such that

$$m(G) < (1+\varepsilon)\, m(E)$$

this is actually just a property of outer measure

Recall that we can write G as a countable union of disjoint open intervals, $G = \bigcup_{j=1}^{\infty} I_j$. Hence

$$m(G) = \sum_{j=1}^{\infty} m(I_j).$$

Define

$$E_j = E \cap I_j$$

it follows that

$$E = \bigcup_{j=1}^{\infty} E_j \quad \underline{\text{and}} \quad m(E) = \sum_{j=1}^{\infty} m(E_j).$$

<u>Claim</u>: $\exists\, j$ such that $m(I_j) < (1+\varepsilon)\, m(E_j)$

and hence $m(E \cap I_j) > \frac{1}{1+\varepsilon} m(I_j)$.

Assuming this Claim we are done (take $\varepsilon = 1/3$). $\square$

[<u>Proof of Claim</u>: Suppose <u>not</u>, then $\sum_j m(I_j) \geq (1+\varepsilon) \sum m(E_j)$

$\Rightarrow m(G) \geq (1+\varepsilon)\, m(E)$.]