

# Review of Riemann Integration (Darboux) [See Abbott, Chap 7]

Let  $[a, b]$  be compact interval &  $f: [a, b] \rightarrow \mathbb{R}$  be bounded.

For each partition  $P$  of  $[a, b]$ , i.e. a finite seq  $\{t_j\}_{j=0}^n$  with

$$a = t_0 < t_1 < \dots < t_n = b$$

we define

$$U(f, P) = \sum_{j=1}^n M_j (t_j - t_{j-1}) \quad [\text{Upper sum}]$$

and

$$L(f, P) = \sum_{j=1}^n m_j (t_j - t_{j-1}) \quad [\text{Lower sum}]$$

where

$$M_j = \sup_{x \in [t_{j-1}, t_j]} f(x) \quad \& \quad m_j = \inf_{x \in [t_{j-1}, t_j]} f(x)$$

$\inf$  &  $\sup$  taken over all partitions  $P$

Then we define

$$U(f) = \inf_P U(f, P) \quad \& \quad L(f) = \sup_P L(f, P)$$

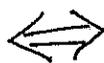
[Upper integral]

[Lower integral]

\*\* If  $U(f) = L(f)$ , then their common value is the Riemann integral of  $f$  is denoted by  $\int_a^b f(x) dx$  &  $f$  is said to be Riemann integrable.

Useful Criterion:

$f: [a, b] \rightarrow \mathbb{R}$  bounded  
is Riemann integrable



$\forall \epsilon > 0 \exists$  partition  $P_\epsilon$  s.t.  
 $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$ .

Can you prove this? (Exercise).

• (First) Theorem

If  $f$  is continuous on  $[a, b]$ , then  $f$  is Riemann integrable on  $[a, b]$ .

\* The proof of this is an easy exercise (do it!), it starts like this:

"Because  $f$  is conts on  $[a, b]$  it is in fact uniformly conts..."

Exercise: Prove this fact.

discontinuous at every pt of  $[a, b]$ !

• Example (Non-Riemann integrable function)

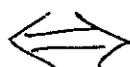
Dirichlet Function:  $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$

(Verification of this should be easy exercise)

$E \subseteq \mathbb{R}$  has "measure zero" if  $\forall \epsilon > 0$   
 $\exists$  countable collection of open intervals  $O_n$  s.t.  
 $E \subseteq \bigcup_{n=1}^{\infty} O_n$  &  $\sum_{n=1}^{\infty} |O_n| < \epsilon$

• (Lebesgue's) Criterion (due to Riemann?)

$f: [a, b] \rightarrow \mathbb{R}$  bounded  
 is Riemann integrable



Set of discontinuities  
 of  $f$  has "measure zero"

Proof (Hard exercise!)

\* Riemann Integral has some nice properties.

Let  $\mathcal{R}$  denote the space of all Riemann integrable fns on  $[a, b]$ .

1.  $\mathcal{R}$  form a vector space over  $\mathbb{R}$  & the integral is a linear functional  
 [i.e.  $f, g \in \mathcal{R}$  &  $\lambda, \mu \in \mathbb{R} \Rightarrow \lambda f + \mu g \in \mathcal{R}$  &  $\int_a^b (\lambda f + \mu g)(x) dx = \lambda \int_a^b f(x) dx + \mu \int_a^b g(x) dx$ ]

2. Fundamental Theorem of Calculus.

3. If  $f_n \rightarrow f$  uniformly on  $[a, b]$  & each  $f_n \in \mathcal{R}$ , then  $f \in \mathcal{R}$  and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b (\lim_{n \rightarrow \infty} f_n(x)) dx.$$

# Why develop a new integral - the Lebesgue integral?

1. Allows us to integrate a larger class of functions.  
(But this not the real reason!)

2.\* The Riemann integral does not behave well under limiting operations when the convergence is not uniform.

Example: Let  $g_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}_n \cap [0,1] \text{ with denominator } \leq n \\ 0 & \text{o/w.} \end{cases}$

then  $g_n \rightarrow g$ , the Dirichlet fn (but not uniformly!) since each  $g_n \in \mathcal{R}$ , but  $g \notin \mathcal{R}$ .

\* Prove this & that  $\int_0^1 g_n = 0 \forall n$ \*

Can you show this directly from defn of uniform conv?

In fact,  $\exists$  seq of conts functions  $\{f_n\}$  converging to  $f$  such that

(i)  $0 \leq f_n(x) \leq 1 \forall x, n$

(ii)  $f_n(x)$  ~~decreasing~~ as  $n \rightarrow \infty$  for all  $x$ . ← Homework Problem

(iii)  $f$  not Riemann integrable !!!

\*\* However, with the Lebesgue integral, which we will construct, the following is true (in fact much more is true):

The class of Lebesgue int'ble fns  $\mathcal{L}$  contains  $\mathcal{R}$

Theorem (Special Case of Dominated Conv. Thm)

Let  $f_n: [a,b] \rightarrow \mathbb{R}$  be seq of Lebesgue int'ble fns with  $|f_n(x)| \leq M \forall x \in [a,b]$  &  $n \in \mathbb{N}$ .

If  $f_n \rightarrow f(x)$  as  $n \rightarrow \infty \forall x \in [a,b]$ , then  $f \in \mathcal{L}$  and  $\int f_n \rightarrow \int f$ .

