Review of Riemann Integration (Darboux) [See Abbott, Chap 7]

Let $[a,b]$ be compact interval & $f:[a,b]\to\mathbb{R}$ be bounded.

For each partition P of $[a,b]$, i.e. a finite seq $\{t_j\}_{j=0}^{n}$ with

$$a = t_0 < t_1 < \cdots < t_n = b$$

we define

$$U(f,P) = \sum_{j=1}^{n} M_j\,(t_j - t_{j-1}) \qquad \text{[Upper sum]}$$

and

$$L(f,P) = \sum_{j=1}^{n} m_j\,(t_j - t_{j-1}) \qquad \text{[Lower sum]}$$

where

$$M_j = \sup_{x\in[t_{j-1},t_j]} f(x) \quad \& \quad m_j = \inf_{x\in[t_{j-1},t_j]} f(x)$$

inf & sup taken over all partitions P

Then we define

$$U(f) = \inf_{P} U(f,P) \quad \& \quad L(f) = \sup_{P} L(f,P)$$

[Upper integral] [Lower integral]

** If $U(f) = L(f)$, then their common value is the Riemann integral of f is denoted by $\int_a^b f(x)\,dx$ & f is said to be Riemann integrable.

Useful Criterion:

$f:[a,b]\to\mathbb{R}$ bounded is Riemann integrable $\iff$ $\forall\, \varepsilon>0\ \exists$ partition P_ε s.t. $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$.

¿ Can you prove this? (Exercise).

- (First) Theorem

If f is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$.

* The proof of this is an easy exercise (do it!), it starts like this: "Because f is cont's on $[a,b]$ it is in fact uniformly cont's...."

Exercise: Prove this fact.

- Example (Non-Riemann integrable function)

discontinuous at every pt of $[0,1]$!

Dirichlet Function: $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$

(Verification of this should be easy exercise)

$E \subseteq \mathbb{R}$ has "measure zero" if $\forall \varepsilon > 0$ $\exists$ countable collection of open intervals O_n s.t. $E \subseteq \bigcup_{n=1}^{\infty} O_n$ & $\sum_{n=1}^{\infty} |O_n| < \varepsilon$

- (Lebesgue's) Criterion (due to Riemann?)

$f: [a,b] \to \mathbb{R}$ bounded is Riemann integrable $\iff$ Set of discontinuities of f has "measure zero"

Proof (Hard exercise!)

* Riemann Integral has some nice properties.

Let R denote the space of all Riemann integrable fns on $[a,b]$.

1. R form a vector space over $\mathbb{R}$ & the integral is a linear functional [i.e. $f, g \in R$ & $\lambda, \mu \in \mathbb{R} \Rightarrow \lambda f + \mu g \in R$ & $\int_a^b (\lambda f + \mu g)(x)\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx$
2. Fundamental Theorem of Calculus.
3. If $f_n \to f$ uniformly on $[a,b]$ & each $f_n \in R$, then $f \in R$ and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b (\lim_{n\to\infty} f_n(x))\,dx$.

Why develop a new integral – the Lebesgue integral?

1. Allows us to integrate a larger class of functions.
   (But this not the real reason!)

2.* The Riemann integral does not behave well under limiting operations when the convergence is not uniform.

Example: Let $g_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \text{ with denominator} \le n \\ 0 & \text{o/w.} \end{cases}$

then $g_n \to g$, the Drichlet fn (but not uniformly!) since each $g_n \in \mathcal{R}$, but $g \notin \mathcal{R}$.

* Prove this & that $\int_0^1 g_n = 0 \; \forall n$ *

¿ Can you show this directly from defn of uniform conv?

In fact, $\exists$ seq of conts functions $\{f_n\}$ converging to f such that

(i) $0 \le f_n(x) \le 1 \quad \forall x, n$

(ii) $f_n(x)$ decreasing as $n \to \infty$ for all x. ← Homework Problem

(iii) f not Riemann integrable !!!

** However, with the Lebesgue integral, which we will construct, the following is true (in fact much more is true):

The class of Lebesgue int'ble fn $\mathcal{L}$ contains $\mathcal{R}$

Theorem (Special Case of Dominated Conv. Thm)

Let $f_n : [a,b] \to \mathbb{R}$ be seq of Lebesgue int'ble fns with $|f_n(x)| \le M \; \forall x \in [0,1]$ & $n \in \mathbb{N}$.

If $f_n(x) \to f(x)$ as $n \to \infty \; \forall x \in [0,1]$, then $f \in \mathcal{L}$ and $\int f_n \to \int f$.