Lebesgue Density Theorem

Given any $E \subseteq \mathbb{R}^n$ measurable,

$$\lim_{r\to 0^+} \frac{m(E\cap B_r(x))}{m(B_r(x))} = 1 \quad \text{for a.e. } x\in E.$$

Lemma (Vitali Covering Lemma)

Given any collection $B_1, \dots, B_N$ of open balls in $\mathbb{R}^n$, $\exists$ disjoint subcollection $\tilde{B}_1, \dots, \tilde{B}_M$ such that $m\left(\bigcup_{j=1}^{N} B_j\right) \le 3^n \sum_{j=1}^{M} m(\tilde{B}_j)$.

Proof: See Lemma 1.2 on p102 of Stein.

Proof We may assume that E is bounded.

Our goal is to show that

$$m_*\left(\left\{x\in E : \liminf_{r\to 0^+} \frac{m(E\cap B_r(x))}{m(B_r(x))} < 1\right\}\right) = 0.$$

It thus suffices to show that $m_*(A_{1/k}) = 0$ $\forall k\in\mathbb{N}$ where

$$A_{1/k} := \left\{x\in E : \liminf_{r\to 0^+} \frac{m(E\cap B_r(x))}{m(B_r(x))} < 1-\frac{1}{k}\right\}.$$

We now fix k, let $A := A_{1/k}$ and proceed to establish that

$$m_*(A) < \varepsilon \quad \forall \varepsilon > 0.$$

Let $\varepsilon>0$. We know $\exists$ open set G with $A\subseteq G$ and

$$m(G) \le m_*(A) + \varepsilon/k.$$

By the definition of A, we know that for every $a\in A$ $\exists$ open ball $B_a \subseteq G$ with (rational) radius r centered at a with

$$m(E\cap B_{r_a}(a)) < \left(1-\frac{1}{k}\right) m(B_{r_a}(a)).$$

* It is not hard to see that if we relax the requirement that this open ball is centered at a, to merely containing a, then we may further assume that each ball is centered at a point in $\mathbb{Q}^n$. (Exercise)

We have thus produced a <u>countable</u> cover $A \subseteq \bigcup_{j=1}^{\infty} B_j$ with each B_j an open ball such that $m(E \cap B_j) < (1 - \frac{1}{k}) m(B_j)$.

By continuity $\exists N$ such that $m_*(A) \leq m\left(\bigcup_{j=1}^{N} B_j\right) + \varepsilon$.

Covering lemma $\Rightarrow$ $\exists$ disjoint subcollection $\tilde{B}_1, \dots, \tilde{B}_M$ of $\{B_j\}_{j=1}^{N}$ such that $m\left(\bigcup_{j=1}^{N} B_j\right) \leq 3^n \sum_{j=1}^{M} m(\tilde{B}_j)$.

$$\Rightarrow \quad m_*(A) \leq 3^n \sum_{j=1}^{M} m(\tilde{B}_j) + \varepsilon.$$

Let $X = \bigcup_{j=1}^{M} \tilde{B}_j$, then

$$m_*(A) \leq m_*(A \cap X) + m_*(A \setminus X)$$

$$\leq m_*(E \cap X) + m_*(G \setminus X)$$

$$= m_*\left(\bigcup_{j=1}^{M} (E \cap \tilde{B}_j)\right) + m(G) - m(X)$$

disjointness

$$= \sum_{j=1}^{M} m(E \cap \tilde{B}_j) + m(G) - m(X)$$

$$\leq \left(1 - \frac{1}{k}\right) \underbrace{\sum_{j=1}^{M} m(\tilde{B}_j)}_{m(X)} + m(G) - m(X)$$

$$= m(G) - \frac{1}{k} m(X)$$

$$\Rightarrow m(X) \leq k \cdot \left(m(G) - m_*(A)\right) < \varepsilon.$$

Thus $m_*(A) \leq (3^n + 1)\varepsilon$. $\square$