Math 8100 Assignment 8
Basic Function Spaces

Due date: Thursday the 2nd of December 2021

1. Prove the following basic properties of $L^\infty = L^\infty(X)$, where X is a measurable subset of $\mathbb{R}^n$:
   (a) $\|\cdot\|_\infty$ is a norm on L^∞ and when equipped with this norm L^∞ is a Banach space.
   (b) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathbb{R}^n$ such that $m(E^c) = 0$ and $f_n \to f$ uniformly on E.
   (c) Simple functions are dense in L^∞, but continuous functions with compact support are not.

   Recall that if $X \subseteq \mathbb{R}^n$ is measurable and f is a measurable function on X, then we define
   $$\|f\|_\infty = \inf\{a \geq 0 : m(\{x \in X : |f(x)| > a\}) = 0\},$$
   with the convention that $\inf \emptyset = \infty$, and
   $$L^\infty = L^\infty(X) = \{f : X \to \mathbb{C} \textit{ measuarable} : \|f\|_\infty < \infty\},$$
   with the usual convention that two functions that are equal a.e. define the same element of L^∞. Thus $f \in L^\infty$ if and only if there is a bounded function g such that $f = g$ almost everywhere; we can take $g = f\chi_E$ where $E = \{x : |f(x)| \leq \|f\|_\infty\}$.

2. Let $X \subseteq \mathbb{R}^n$ be measurable.
   (a) i. Prove that if $m(X) < \infty$, then
   $$L^\infty(X) \subset L^2(X) \subset L^1(X) \tag{1}$$
   with strict inclusion in each case, and that for any measurable $f : X \to \mathbb{C}$ one in fact has
   $$\|f\|_{L^1(X)} \leq m(X)^{1/2}\|f\|_{L^2(X)} \leq m(X)\|f\|_{L^\infty(X)}.$$
   ii. Give examples to show that no such result of the form (1) can hold if one drops the assumption that $m(x) < \infty$. Prove, furthermore, that if $L^2(X) \subseteq L^1(X)$, then $m(X) < \infty$.
   (b) Prove that
   $$\underbrace{L^1(X) \cap L^\infty(X) \subset L^2(X)}_{(\star)} \subset L^1(X) + L^\infty(X)$$
   and that in addition to $(\star)$ one in fact has
   $$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2}\|f\|_{L^\infty(X)}^{1/2}$$
   for any measurable function $f : X \to \mathbb{C}$.

3. Prove that
   $$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$$
   with strict inclusion in each case, and that for any sequence $a = \{a_j\}_{j\in\mathbb{Z}}$ of complex numbers one in fact has
   $$\|a\|_{\ell^\infty(\mathbb{Z})} \leq \|a\|_{\ell^2(\mathbb{Z})} \leq \|a\|_{\ell^1(\mathbb{Z})}.$$

   Recall that for $p = 1, 2, \infty$ we define
   $$\ell^p(\mathbb{Z}) = \{a = \{a_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{C} : \|a\|_{\ell^p(\mathbb{Z})} < \infty\}$$
   where
   $$\|a\|_{\ell^1(\mathbb{Z})} = \sum_{j=-\infty}^{\infty} |a_j|, \quad \|a\|_{\ell^2(\mathbb{Z})} = \Big(\sum_{j=-\infty}^{\infty} |a_j|^2\Big)^{1/2}, \textit{ and } \|a\|_{\ell^\infty(\mathbb{Z})} = \sup_j |a_j|.$$

4. Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

   (a) Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

   (b) Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$

5. Let H be a Hilbert space with orthonormal basis $\{u_n\}_{n=1}^{\infty}$.

   (a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Prove that

   $$\sum_{n=1}^{\infty} a_n u_n \text{ converges in } H \iff \sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

   and moreover that if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, then $\left\|\sum_{n=1}^{\infty} a_n u_n\right\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$.

   (b) i. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

   ii. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1/2}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

6. For each $1 \le p \le \infty$, define $\Lambda_p : L^p([0,1]) \to \mathbb{R}$ by

   $$\Lambda_p(f) = \int_0^1 x^2 f(x)\,dx.$$

   Explain why Λ_p is a continuous linear functional and compute its norm (in terms of p).

Extra Practice Problems

Not to be handed in with the assignment

1. Let f and g be two non-negative Lebesgue measurable functions on $[0,\infty)$. Suppose that

   $$A := \int_0^{\infty} f(y)\,y^{-1/2} dy < \infty \qquad \text{and} \qquad B := \left(\int_0^{\infty} |g(y)|^2 dy\right)^{1/2} < \infty$$

   Prove that

   $$\int_0^{\infty} \left(\int_0^x f(y)\,dy\right) \frac{g(x)}{x}\,dx \le AB$$

2. Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \le 1$ for all $k \in \mathbb{N}$.

   (a) i. Prove that if $f_k \to f$ either a.e. on $[0,1]$ or in $L^1([0,1])$, then $f \in L^2([0,1])$ with $\|f\|_2 \le 1$.

   ii. Do either of the above hypotheses guarantee that $f_k \to f$ in $L^2([0,1])$?

   (b) Prove that if $f_k \to f$ a.e. on $[0,1]$, then this in fact implies that $f_k \to f$ in $L^1([0,1])$.

3. Let $1 \le p \le \infty$. Prove that if $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $L^p(\mathbb{R}^n)$ with the property that

   $$\sum_{k=1}^{\infty} \|f_k\|_p < \infty,$$

   then $\sum f_k$ converges almost everywhere to an $L^p(\mathbb{R}^n)$ function with

   $$\left\|\sum_{k=1}^{\infty} f_k\right\|_p \le \sum_{k=1}^{\infty} \|f_k\|_p.$$