Math 8100 Assignment 5
Repeated Integration

Due date: Tuesday the 26th of October 2021

1. Let $f \in L^1([0,1])$, and for each $x \in [0,1]$ define
$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt.$$
Show that $g \in L^1([0,1])$ and that
$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx.$$

2. Carefully prove that if we define
$$f(x,y) := \begin{cases} \dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$
for each $(x,y) \in \mathbb{R}^2$, then f defines a function in $L^1(\mathbb{R}^2)$.

3. Prove that if $f \in L^1(\mathbb{R}^n)$, then
$$\int_{\mathbb{R}^n} |f(x)|\,dx = \int_0^\infty m(\{x \in \mathbb{R}^n \,:\, |f(x)| > t\})\,dt.$$

4. Let $A, B \subseteq \mathbb{R}^n$ be bounded measurable sets with positive Lebesgue measure. For each $t \in \mathbb{R}^n$ define the function
$$g(t) = m\left(A \cap (t - B)\right)$$
where $t - B = \{t - b \,:\, b \in B\}$.
   (a) Prove that g is a continuous function and
   $$\int_{\mathbb{R}^n} g(t)\,dt = m(A)\,m(B).$$
   (b) Conclude that the sumset
   $$A + B = \{a + b \,:\, a \in A \text{ and } b \in B\}$$
   contains a non-empty open subset of $\mathbb{R}^n$.

5. Let $f, g \in L^1([0,1])$ and for each $0 \le x \le 1$ define
$$F(x) := \int_0^x f(y)\,dy \quad \text{and} \quad G(x) := \int_0^x g(y)\,dy.$$
Prove that
$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$

6. Suppose that F is a closed subset of $\mathbb{R}$ whose complement has finite measure. Let $\delta(x)$ denote the distance from x to F, namely

$$\delta(x) = d(x, F) = \inf\{|x - y| \,:\, y \in F\}$$

and

$$I_F(x) = \int_{-\infty}^{\infty} \frac{\delta(y)}{|x - y|^2}\, dy.$$

   (a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition $|\delta(x)-\delta(y)| \le |x-y|$.
   (b) Show that $I_F(x) = \infty$ if $x \notin F$.
   (c) Show that $I_F(x) < \infty$ for a.e. $x \in F$, by showing that $\int_F I_F(x)\, dx < \infty$.

7. Let $f \in L^1(\mathbb{R})$. For any $h > 0$ we define

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy$$

   (a) Prove that for all $h > 0$,

$$\int_{\mathbb{R}} |A_h(f)(x)|\, dx \le \int_{\mathbb{R}} |f(x)|\, dx.$$

   (b) Prove that

$$\lim_{h \to 0^+} \int_{\mathbb{R}} |A_h(f)(x) - f(x)|\, dx = 0.$$

One can in fact show that $\lim_{h\to 0^+} A_h(f) = f$ almost everywhere. This result is actually equivalent to the Lebesgue Density Theorem in $\mathbb{R}$ and we will establish this later in the course.

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that if $\{a_{jk}\}_{(j,k)\in\mathbb{N}\times\mathbb{N}}$ is a "double sequence" with $a_{jk} \ge 0$ for all $(j,k) \in \mathbb{N}\times\mathbb{N}$, then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sup\Big\{ \sum_{(j,k)\in B} a_{jk} \,:\, B \text{ is a finite subset of } \mathbb{N}\times\mathbb{N}\Big\}$$

and deduce from this that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk}.$$

This conclusion holds more generally provided $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |a_{jk}| < \infty$, see Theorem 8.3 in "Baby Rudin".

2. (a) Prove that

$$\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx = \infty.$$

   (b) By considering the iterated integral

$$\int_0^{\infty}\left(\int_0^{\infty} x e^{-xy}(1 - \cos y)\, dy\right) dx$$

   show (with justification) that

$$\lim_{A\to\infty} \int_0^{A} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$