

Math 8100 Assignment 3
Lebesgue measurable sets and functions

Due date: Thursday the 23rd of September 2021

1. Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $m(E) < \infty$, then for every $\varepsilon > 0$ there exists a set A that is a finite union of closed cubes such that $m(E \Delta A) < \varepsilon$.

[Recall that $E \Delta A$ stands for the symmetric difference, defined by $E \Delta A = (E \setminus A) \cup (A \setminus E)$]

2. A set $E \subseteq \mathbb{R}^n$ is said to be *Carathéodory measurable* if $m_*(A) = m_*(A \cap E) + m_*(E^c \cap A)$ for every set $A \subseteq \mathbb{R}^n$, where m_* denotes Lebesgue outer measure.

Prove that a set $E \subseteq \mathbb{R}^n$ is *Lebesgue measurable* if and only if E is *Carathéodory measurable*.

Hint: If E is Lebesgue measurable and A is any set, choose a G_δ set $B \supseteq A$ with $m_(A) = m(B)$. Conversely, if E is Carathéodory measurable and $m_*(E) < \infty$, choose a G_δ set B with $B \supseteq E$ with $m_*(E) = m(B)$ and conclude that $m_*(B \setminus E) = 0$. Be sure to also consider the case when $m_*(E) = \infty$.*

3. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *upper semicontinuous* at a point x in \mathbb{R}^n if

$$f(x) \geq \limsup_{y \rightarrow x} f(y).$$

Prove that if f is upper semicontinuous at every point x in \mathbb{R}^n , then f is Borel measurable.

4. Let $\{f_n\}$ be a sequence of measurable functions on \mathbb{R}^n . Prove that $\{x \in \mathbb{R}^n : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$ defines a measurable set.

5. Recall that the **Cantor set** \mathcal{C} is the set of all $x \in [0, 1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k . Consider the function

$$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where } b_k = a_k/2.$$

- (a) Show that f is well defined and continuous on \mathcal{C} , and moreover $f(0) = 0$ as well as $f(1) = 1$.
(b) Prove the existence of a continuous function that maps a measurable set to a non-measurable set.

6. Let us examine the map f defined in Question 5 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0, 1]$ to obtain \mathcal{C} . In this case $f(x) = \ell 2^m$ for some integers ℓ and m , and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0, 1] \rightarrow [0, 1]$ by declaring it to be constant on each interval missing from \mathcal{C} . F is called the **Cantor-Lebesgue function**.

- (a) Prove that F is non-decreasing and continuous.
(b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0, 1]$ to $[0, 2]$.
(c) i. Show that $m(G(\mathcal{C})) = 1$.
ii. By considering rational translates of \mathcal{N} (the non-measurable subset of $[0, 1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set \mathcal{N}' .
iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.
(d) Give an example of a measurable function φ such that $\varphi \circ G^{-1}$ is not measurable.

Hint: Let φ be the characteristic function of a null set whose image under G is not measurable.

Extra Challenge Problems

Not to be handed in with the assignment

1. Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no function f satisfying $f = \chi_{[0,1]}$ almost everywhere which is also continuous on all of \mathbb{R} .
2. Question 6d above supplies us with an example that if f and g are Lebesgue measurable, then it does not necessarily follow that $f \circ g$ will be Lebesgue measurable, even if g is assumed to be continuous.
Prove that if f is Borel measurable, then $f \circ g$ will be Lebesgue or Borel measurable whenever g is.
3. Let f be a measurable function on $[0, 1]$ with $|f(x)| < \infty$ for a.e. x . Prove that there exists a sequence of continuous functions $\{g_n\}$ on $[0, 1]$ such that $g_n \rightarrow f$ for a.e. $x \in [0, 1]$.