Math 8100 Assignment 1
Preliminaries

Due date: Thursday the 2nd of September 2021

1. The **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Thus $\mathcal{C}$ is obtained from $[0,1]$ by removing the open middle third $(\frac{1}{3}, \frac{2}{3})$, then removing the open middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the two remaining intervals, and so forth.
   (a) Find a real number x belonging to the Cantor set which is not the endpoint of one of the intervals used in its construction.
   (b) Prove that $\mathcal{C}$ is both nowhere dense (and hence meager) and has measure zero.
   (c) Prove that $\mathcal{C}$ is uncountable by showing that the function $f(x) = \sum_{k=1}^{\infty} b_k 2^{-k}$ where $b_k = a_k/2$, maps $\mathcal{C}$ onto $[0,1]$.

2. A set $A \subseteq \mathbb{R}^n$ is called an F_σ set if it can be written as the countable union of closed subsets of $\mathbb{R}^n$. A set $B \subseteq \mathbb{R}^n$ is called a G_δ set if it can be written as the countable intersection of open subsets of $\mathbb{R}^n$.
   (a) Argue that a set is a G_δ set if and only if its complement is an F_σ set.
   (b) Show that every closed set is a G_δ set and every open set is an F_σ set.
   Hint: One approach is to prove that every open subset of $\mathbb{R}^n$ can be written as a countable union of closed cubes with disjoint interiors. This approach is however very specific to open sets in $\mathbb{R}^n$.
   (c) Give an example of an F_σ set which is not a G_δ set and a set which is neither an F_σ nor a G_δ set.

3. (a) Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of all the rationals in $[0,1]$ and define $f : [0,1] \to \mathbb{R}$ by setting
   $$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases} .$$
   Prove that $\lim_{x \to c} f(x) = 0$ for every $c \in [0,1]$ and conclude that set of all points at which f is discontinuous is precisely $[0,1] \cap \mathbb{Q}$.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be bounded.
      i. Recall that we defined the *oscillation of f at x* to be
      $$\omega_f(x) := \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|.$$
      Briefly explain why this is a well defined notion and prove that
      $$f \text{ is continuous at } x \iff \omega_f(x) = 0.$$
      ii. Prove that for every $\varepsilon > 0$ the set $A_\varepsilon = \{x \in \mathbb{R} : \omega_f(x) \geq \varepsilon\}$ is closed and deduce from this that the set of all points at which f is discontinuous is an F_σ set.

4. Let $\{x_n\}_{n=1}^{\infty}$ be any enumeration of a given countable set $X \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$ define
   $$f_n(x) = \begin{cases} 1 \text{ if } x > x_n \\ 0 \text{ if } x \leq x_n \end{cases} .$$
   Prove that
   $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x)$$
   defines an increasing function f on $\mathbb{R}$ that is continuous on $\mathbb{R} \setminus X$.

5. Let $C([0,1])$ denote the collection of all real-valued continuous functions with domain $[0,1]$.
   (a) Show that $d_\infty(f,g) = \sup\limits_{x\in[0,1]} |f(x)-g(x)|$ defines a metric on $C([0,1])$ and that with the "uniform" metric $C([0,1])$ is in fact a *complete* metric space.
   (b) Prove that the unit ball $\{f \in C([0,1]) : d_\infty(f,0) \leq 1\}$ is closed and bounded, but *not* compact.
   (c) ** Challenge: Can you show that $C([0,1])$ with the metric d_∞ is not *totally bounded.*
   A set is totally bounded if, for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε.

6. Let

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}.$$

   (a) Show that the series defining g does not converge uniformly on $(0,\infty)$, but none the less still defines a continuous function on $(0,\infty)$.
   Hint for the first part: Show that if $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set X, then the sequence of functions $\{g_n\}$ must converge uniformly to 0 on X.
   (b) Is g differentiable on $(0,\infty)$? If so, is the derivative function g' continuous on $(0,\infty)$?

7. Let $h_n(x) = \dfrac{x}{(1+x)^{n+1}}$.
   (a) Prove that h_n converges uniformly to 0 on $[0,\infty)$.
   (b) i. Verify that

$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$

   ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0,\infty)$?
   (c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a,\infty)$ for any $a > 0$.

Extra Challenge Problems

Not to be handed in with the assignment

1. Given an arbitrary F_σ set V, can you produce a function whose discontinuities lie precisely in V?
   Hint: First try to do this for an arbitrary closed set.

2. (Baire Category Theorem) Prove that if X is a non-empty *complete* metric space, then X cannot be written as a countable union of nowhere dense sets.

   Hint: Modify the proof given in class of the special case $X = \mathbb{R}$ replacing the use of the nested interval property with the following fact (which you should prove):

   If $F_1 \supseteq F_2 \supseteq \cdots$ is a nested sequence of closed non-empty and bounded sets in a complete metric space X with $\lim\limits_{n\to\infty} \operatorname{diam} F_n = 0$, then $\bigcap\limits_{n=1}^{\infty} F_n$ contains exactly one point.

3. Complete the proof, sketched in class, of the so-called Lebesgue Criterion: *A bounded function on an interval $[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.*
   (a) Prove that if the set of discontinuities of f has measure zero, then f is Riemann integrable.
   [*Hint: Let $\varepsilon > 0$. Cover the compact set A_ε (defined in Q3(b)ii. above) by a finite number of open intervals whose total length is $\leq \varepsilon$. Select and appropriate partition of $[a,b]$ and estimate the difference between the upper and lower sums of f over this partition.*]
   (b) Prove that if f is Riemann integrable on $[a,b]$, then its set of discontinuities has measure zero.
   [*Hint: The set of discontinuities of f is contained in $\bigcup_n A_{1/n}$. Given $\varepsilon > 0$, choose a partition P such that $U(f,P) - L(f,P) < \varepsilon/n$. Show that the total length of the intervals in P whose interiors intersect $A_{1/n}$ is $\leq \varepsilon$.*]