Math 8100 Exam 1

Thursday the 14th of October 2021

Answer any <u>THREE</u> of the following four problems

1. (a) Prove, using the definition of Lebesgue outer measure, that $m_*\big(\bigcup_{j=1}^{\infty} E_j\big) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

   (b) Let $E_1, E_2, \ldots$ be a countable collection of Lebesgue measurable subsets of $\mathbb{R}^n$ that increase to E in the sense that $E_j \subseteq E_{j+1}$ for all j, and $E = \bigcup_{j=1}^{\infty} E_j$. Use the countable additivity of Lebesgue measure to prove that
   $$m(E) = \lim_{j\to\infty} m(E_j).$$

   (c) Prove that for any Lebesgue measurable set $E \subseteq \mathbb{R}^n$ one has
   $$m(E) = \sup\{m(K) \,:\, K \subseteq E \text{ with } K \text{ compact}\}.$$
   Hint: You may use, without proof, the fact that for any Lebesgue measurable set $E \subseteq \mathbb{R}^n$ and any $\varepsilon > 0$ there exists a closed set F with $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon$.

2. (a) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then for any $\delta > 0$ the dilated set
   $$\delta E := \{\delta x \,:\, x \in E\}$$
   is also Lebesgue measurable and satisfies $m(\delta E) = \delta^n m(E)$.

   (b) Carefully state Fatou's Lemma and deduce the Monotone Convergence Theorem from it.

   (c) Prove that if f is a non-negative measurable function on $\mathbb{R}^n$ and $\delta > 0$, then f_δ, defined by
   $$f_\delta(x) = \delta^{-n} f(\delta^{-1}x)$$
   is also a non-negative measurable function and
   $$\int f(x)\,dx = \int f_\delta(x)\,dx.$$

3. (a) Carefully state the Dominated Convergence Theorem.

   (b) Let $f \in L^1(\mathbb{R})$ and define
   $$F(t) = \int_{\mathbb{R}} f(x)\sin(tx)\,dx.$$
   i. Prove that F is a bounded continuous function on $\mathbb{R}$.

   ii. Prove that if one additionally assumes that $xf(x)$ is a Lebesgue integrable function on $\mathbb{R}$, then F is in fact differentiable at every t and find a formula for $F'(t)$.

4. (a) Prove that if $f \in L^1(\mathbb{R}^n)$, then $|f(x)| < \infty$ almost everywhere.

   (b) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions in $L^1(\mathbb{R}^n)$.

   i. Prove that if $\sum\limits_{k=1}^{\infty} \|f_k\|_1 < \infty$, then $\sum\limits_{k=1}^{\infty} f_k(x) < \infty$ almost everywhere.

   Hint: You may use, without proof, the fact that for any sequence $\{f_k\}_{k=1}^{\infty}$ in $L^+(\mathbb{R}^n)$ one has
   $$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k.$$

   ii. Prove that if $f_k \to f$ in L^1 then $\{f_k\}_{k=1}^{\infty}$ must contain a subsequence which converges to f almost everywhere.