The Problem of Measure

Question: Can we assign a "measure" to all subsets of $\mathbb{R}^n$ that generalizes the usual notion of volume in $\mathbb{R}^n$ (that we understand for "elementery sets", such as cubes)?

Such a thing should of course be a function

$$m : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$$

we should agree with this, right?

power set of $\mathbb{R}^n$, the set of all subsets of $\mathbb{R}^n$.

and should further satisfy the following:

(a) If $E_1, E_2, \ldots$ is a finite or infinite sequence of disjoint sets, then

$$m(E_1 \cup E_2 \cup \cdots) = m(E_1) + m(E_2) + \cdots$$

(b) If E is congruent to F (E can be transformed into F by translation, rotation & reflection), then $m(E) = m(F)$

(c) $m(Q) = 1$, where $Q = \{x \in \mathbb{R}^n : 0 \le x_j < 1 \ (1 \le j \le n)\}$.

(a) is called countable additivity & implies that $m(F) \le m(E)$ whenever $F \subseteq E$, since $E = F \cup (E \setminus F)$.

Answer: NO!!!

this forms a σ-algebra

** However, it is a fundamental result that there does exist a unique measure, Lebesgue measure, provided one restricts oneself to a subclass of "reasanable" sets. **

Existence of "non-measurable" sets

Let us see why the answer to our original question is NO in the case $n=1$.
Define the following equivalence relation among the reals in $[0,1)$:

$$x \sim y \iff x-y \in \mathbb{Q}.$$

Note:
- Two equivalence classes are either the same or disjoint.
- Since each equivalence class is countable (in 1-1 correspondence with $\mathbb{Q}$) there must be uncountably many classes.

Let N be a set which contains exactly one element from each equivalence class. (Note that to do this one must invoke the Axiom of Choice)

Let $\{q_j\}_{j=1}^{\infty}$ be any enumeration of $\mathbb{Q} \cap [-1,1]$ and define

$$N_j := N + q_j$$

Note: $N_j \cap N_k = \phi$ whenever $j \neq k$. (why?)

Now suppose $m : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ satisfies (a), (b) & (c):

- (b) $\Rightarrow m(N_j) = m(N) \ \forall j$
- Since $[0,1) \subseteq \bigcup_{j=1}^{\infty} N_j \subseteq [-1,2)$ it follows from (a) & (c) that

$$1 \leq \sum_{j=1}^{\infty} m(N_j) \leq 3 \ ↯$$

This is a contradiction. If $m(N)=0$, then $\sum_{j=1}^{\infty} m(N_j) = 0$, while if $m(N)>0$, then $\sum_{j=1}^{\infty} m(N_j) = \infty$!!

Faced with this alarming situation one might be tempted to weaken (a) so that additivity is only required to hold for finite seqs.

This is a bad idea in general!

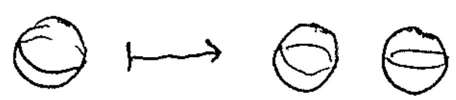
$\longrightarrow$ It is this property for countable sequences that makes all the limit and continuity results in the theory work!

Even more alarming is that when $n \geq 3$, (a) for finite sequences and (b) are inconsistant!!

<u>The Banach–Tarski Paradox</u>:

If X and Y are arbitrary bounded open sets in $\mathbb{R}^n$, $n \geq 3$, then $\exists$ $k \in \mathbb{N}$ and partition $\{X_j : 1 \leq j \leq k\}$ & $\{Y_j : 1 \leq j \leq k\}$ of X & Y such that X_j is congruent to Y_j for all j.

Thus one can cut up a ball the size of a pea into a finite number of pieces & rearrange them to form a ball the size of the earth!

[ only requires 5 pieces!!!]