

Math 8100 Assignment 7

Hilbert Spaces

Due date: Thursday 14th of November 2019

1. (a) Prove that $\ell^2(\mathbb{N})$ is complete.

Recall that $\ell^2(\mathbb{N}) := \{x = \{x_j\}_{j=1}^\infty : \|x\|_{\ell^2} < \infty\}$, where $\|x\|_{\ell^2} := \left(\sum_{j=1}^\infty |x_j|^2\right)^{1/2}$.

- (b) Let H be a Hilbert space. Prove the so-called *polarization identity*, namely that for any $x, y \in H$,

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

and conclude that any invertible linear map from H to $\ell^2(\mathbb{N})$ is *unitary* if and only if it is *isometric*.

*Recall that if H_1 and H_2 are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, then a mapping $U : H_1 \rightarrow H_2$ is said to be **unitary** if it is an invertible linear map that preserves inner products, namely $\langle Ux, Uy \rangle_2 = \langle x, y \rangle_1$, and an **isometry** if it preserves “lengths”, namely $\|Ux\|_2 = \|x\|_1$.*

2. Let E be a subset of a Hilbert space H .

(a) Show that $E^\perp := \{x \in H : \langle x, y \rangle = 0 \text{ for all } y \in E\}$ is a closed subspace of H .

(b) Show that $(E^\perp)^\perp$ is the smallest closed subspace of H that contains E .

3. In $L^2([0, 1])$ let $e_0(x) = 1$, $e_1(x) = \sqrt{3}(2x - 1)$ for all $x \in (0, 1)$.

(a) Show that e_0, e_1 is an orthonormal system in $L^2(0, 1)$.

(b) Show that the polynomial of degree 1 which is closest with respect to the norm of $L^2(0, 1)$ to the function $f(x) = x^2$ is given by $g(x) = x - 1/6$. What is $\|f - g\|_2$?

4. (a) Verify that the following systems are orthogonal in $L^2([0, 1])$:

i. $\{1/\sqrt{2}, \cos(2\pi x), \sin(2\pi x), \dots, \cos(2\pi kx), \sin(2\pi kx), \dots\}$

ii. $\{e^{2\pi i k x}\}_{k=-\infty}^\infty$

(b) Let $f \in L^1([0, 1])$.

i. Show that for any $\epsilon > 0$ we can write $f = g + h$, where $g \in L^2$ and $\|h\|_1 < \epsilon$.

ii. Use this decomposition of f to prove the so-called *Riemann-Lebesgue lemma*:

$$\lim_{k \rightarrow \infty} \int_0^1 f(x) \cos(2\pi kx) dx = \lim_{k \rightarrow \infty} \int_0^1 f(x) \sin(2\pi kx) dx = 0$$

5. (a) The first three Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2 - 1)/2.$$

Show that the orthonormal system in $L^2([-1, 1])$ obtained by applying the Gram-Schmidt process to $1, x, x^2$ are scalar multiples of these.

(b) Compute

$$\min_{a,b,c} \int_{-1}^1 |x^3 - a - bx - cx^2|^2 dx$$

(c) Find

$$\max \int_{-1}^1 x^3 g(x) dx$$

where g is subject to the restrictions

$$\int_{-1}^1 g(x) dx = \int_{-1}^1 xg(x) dx = \int_{-1}^1 x^2g(x) dx = 0; \quad \int_{-1}^1 |g(x)|^2 dx = 1.$$

6. Let

$$\mathcal{C} = \left\{ f \in L^2([0, 1]) : \int_0^1 f(x) dx = 1 \text{ and } \int_0^1 xf(x) dx = 2 \right\}$$

(a) Let $g(x) = 18x^2 - 5$. Show that $g \in \mathcal{C}$ and that

$$\mathcal{C} = g + \mathcal{S}^\perp$$

where \mathcal{S}^\perp denotes the orthogonal complement of $\mathcal{S} = \text{Span}(\{1, x\})$.

(b) Find *the* function $f_0 \in \mathcal{C}$ for which

$$\int_0^1 |f_0(x)|^2 dx = \inf_{f \in \mathcal{C}} \int_0^1 |f(x)|^2 dx.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that every closed convex set K in a Hilbert space has a unique element of minimal norm.

2. **The Mean Ergodic Theorem:** Let U be a unitary operator on a Hilbert space H .

Prove that if $M = \{x : Ux = x\}$ and $S_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n$, then $\lim_{N \rightarrow \infty} \|S_N x - Px\| = 0$ for all $x \in H$, where Px denotes the orthogonal projection of x onto M .