Math 8100 Assignment 6
The Fourier Transform

Due date: Thursday the 31st of October 2019

Recall that we have defined the Fourier transform of an integrable function f on $\mathbb{R}^n$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

where $x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n$ and the convolution of two integrable functions f and g on $\mathbb{R}^n$ by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, dy.$$

1. Prove that if $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$. (This is called the Riemann-Lebesgue lemma.)
   Hint: Write $\widehat{f}(\xi) = \frac{1}{2} \int [f(x) - f(x - \xi')] e^{-2\pi i x \cdot \xi} \, dx$, where $\xi' = \frac{\xi}{2|\xi|^2}$.

2. (a) Prove that if $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi)$ for all $\xi \in \mathbb{R}^n$.
   (b) Conclude from part (a) that
      i. if $f, g, h \in L^1(\mathbb{R}^n)$, then $f * g = g * f$ and $(f * g) * h = f * (g * h)$ almost everywhere.
      ii. there does not exist $I \in L^1(\mathbb{R}^n)$ such that $f * I = f$ almost everywhere for all $f \in L^1(\mathbb{R}^n)$.

3. Let $f \in L^1(\mathbb{R}^n)$.
   (a) Show that if $y \in \mathbb{R}^n$ and
      i. $g(x) = f(x - y)$ for all $x \in \mathbb{R}^n$, then $\widehat{g}(\xi) = e^{-2\pi i y \cdot \xi} \widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$.
      ii. $h(x) = e^{2\pi i x \cdot y} f(x)$ for all $x \in \mathbb{R}^n$, then $\widehat{h}(\xi) = \widehat{f}(\xi - y)$ for all $\xi \in \mathbb{R}^n$.
   (b) Show that if T be a non-singular linear transformation of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ denote its inverse transpose, then

   $$\widehat{f \circ T}(\xi) = \frac{1}{|\det T|} \widehat{f}(S\xi)$$

   for all $\xi \in \mathbb{R}^n$.

4. (a) Let $f \in L^1(\mathbb{R})$.
      i. Let $g(x) = x f(x)$. Show that if $g \in L^1$, then $\widehat{f}$ is differentiable and $\frac{d}{d\xi} \widehat{f}(\xi) = -2\pi i \, \widehat{g}(\xi)$.
      ii. Let $f \in C_0^1(\mathbb{R})$ and $h(x) = \frac{d}{dx} f(x)$. Show that if $h \in L^1$, then $\widehat{h}(\xi) = 2\pi i \xi \widehat{f}(\xi)$.
         Recall that $C_0^1(\mathbb{R})$ is the collection of functions in $C^1(\mathbb{R})$ which vanishes at infinity.
   (b) Let $G(x) = e^{-\pi x^2}$. By considering the derivative of $\widehat{G}(\xi)/G(\xi)$, show that $\widehat{G}(\xi) = G(\xi)$.
      Hint: You may also want to use the fact that $\int_{\mathbb{R}} G(x) \, dx = 1$ (see "challenge" problem).

5. The functions D, F, and P defined below are all bounded $L^+(\mathbb{R})$ functions with integrals equal to 1.
   (a) Show that if

   $$D(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

   then

   $$\widehat{D}(\xi) = \frac{\sin \pi \xi}{\pi \xi}.$$

   This gives, in light of Assignment 5 Challenge Problem 1(a), an explicit example of a function which is not in $L^1(\mathbb{R})$, but yet is the Fourier transform of an L^1 function. See Question 6 for additional higher dimensional examples.

(b) Let

$$F(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

i. Show that

$$\widehat{F}(\xi) = \left(\frac{\sin \pi \xi}{\pi \xi}\right)^2.$$

Hint: It may help to write $\widehat{F}(\xi) = h(\xi) + h(-\xi)$ where $h(\xi) = e^{2\pi i \xi} \int_0^1 y e^{-2\pi i y \xi} dy$.

ii. Find the Fourier transform of the function

$$f(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

Be careful to fully justify your answer.

(c) Show that if

$$P(x) = \frac{1}{\pi} \frac{1}{1 + x^2}.$$

then

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|} e^{2\pi i x \xi} d\xi = P(x)$$

and hence that

$$\widehat{P}(\xi) = e^{-2\pi|\xi|}.$$

Be careful to fully justify your answer.

Remark: *In Questions 4b and 5 above D is for Dirichlet, F is for Fejér, P is for Poisson, and G is for Gauss-Weierstrass. The respective "approximate identities", namely $\{(\widehat{D})_t\}_{t>0}$, $\{(\widehat{F})_t\}_{t>0}$, $\{P_t\}_{t>0}$, and $\{G_{\sqrt{t}}\}_{t>0}$, are generally referred to as Dirichlet, Fejér, Poisson, and Gauss-Weierstrass kernels.*

6. Show that for any $\varepsilon > 0$ the function $F(\xi) = (1 + |\xi|^2)^{-\varepsilon}$ is the Fourier transform of an $L^1(\mathbb{R}^n)$ function.

Hint: Consider the function

$$f(x) = \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} dt,$$

where $G_t(x) = t^{-n} e^{-\pi|x|^2/t^2}$. Now use Fubini/Tonelli to prove that $f \in L^1(\mathbb{R}^n)$ with $\widehat{f}(\xi) = F(\xi)\|f\|_1$.

Extra Challenge Problems
Not to be handed in with the assignment

1. By considering the iterated integral

$$\int_0^\infty \left(\int_0^\infty x e^{-x^2(1+y^2)} dx\right) dy$$

show (with justification) that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

and hence that

$$\int_{-\infty}^\infty e^{-\pi x^2} dx = 1.$$