

Math 8100 Assignment 3

Lebesgue measurable sets and functions

Due date: 5:00 pm Friday the 20th of September 2019

1. (a) Prove that for every $E \subseteq \mathbb{R}^n$ there exists a Borel set $B \supseteq E$ with the property that $m(B) = m_*(E)$.
- (b) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then there exists a Borel set $B \subseteq E$ with the property that $m(B) = m(E)$.
- (c) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $m(E) < \infty$, then for every $\varepsilon > 0$ there exists a set A that is a finite union of closed cubes such that $m(E \Delta A) < \varepsilon$.

[Recall that $E \Delta A$ stands for the symmetric difference, defined by $E \Delta A = (E \setminus A) \cup (A \setminus E)$]

2. Let E be a Lebesgue measurable subset of \mathbb{R}^n with $m(E) > 0$ and $\varepsilon > 0$.
 - (a) Prove that E “almost” contains a closed cube in the sense that there exists a closed cube Q such that $m(E \cap Q) \geq (1 - \varepsilon)m(Q)$.
 - (b) Prove that the so-called difference set $E - E := \{d : d = x - y \text{ with } x, y \in E\}$ necessarily contains an open ball centered at the origin.

Hint: It may be useful to observe that $d \in E - E \iff E \cap (E + d) \neq \emptyset$.

3. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *upper semicontinuous* at a point x in \mathbb{R}^n if

$$f(x) \geq \limsup_{y \rightarrow x} f(y).$$

Prove that if f is upper semicontinuous at every point x in \mathbb{R}^n , then f is Borel measurable.

4. Let $\{f_n\}$ be a sequence of measurable functions on \mathbb{R}^n . Prove that $\{x \in \mathbb{R}^n : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$ defines a measurable set.
5. Recall that the **Cantor set** \mathcal{C} is the set of all $x \in [0, 1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k . Consider the function

$$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where } b_k = a_k/2.$$

- (a) Show that f is well defined and continuous on \mathcal{C} , and moreover $f(0) = 0$ as well as $f(1) = 1$.
 - (b) Prove that there exists a continuous function that maps a measurable set to a non-measurable set.
6. Let us examine the map f defined in Question 5 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0, 1]$ to obtain \mathcal{C} . In this case $f(x) = \ell 2^m$ for some integers ℓ and m , and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0, 1] \rightarrow [0, 1]$ by declaring it to be constant on each interval missing from \mathcal{C} . F is called the **Cantor-Lebesgue function**.
 - (a) Prove that F is non-decreasing and continuous.
 - (b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0, 1]$ to $[0, 2]$.
 - (c)
 - i. Show that $m(G(\mathcal{C})) = 1$.
 - ii. By considering rational translates of \mathcal{N} (the non-measurable subset of $[0, 1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set \mathcal{N}' .
 - iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.
 - (d) Give an example of a measurable function φ such that $\varphi \circ G^{-1}$ is not measurable.

Hint: Let φ be the characteristic function of a null set whose image under G is not measurable.

Extra Challenge Problems

Not to be handed in with the assignment

1. Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no function f satisfying $f = \chi_{[0,1]}$ almost everywhere which is also continuous on all of \mathbb{R} .
2. Question 6d above supplies us with an example that if f and g are Lebesgue measurable, then it does not necessarily follow that $f \circ g$ will be Lebesgue measurable, even if g is assumed to be continuous.
Prove that if f is Borel measurable, then $f \circ g$ will be Lebesgue or Borel measurable whenever g is.
3. Let f be a measurable function on $[0, 1]$ with $|f(x)| < \infty$ for a.e. x . Prove that there exists a sequence of continuous functions $\{g_n\}$ on $[0, 1]$ such that $g_n \rightarrow f$ for a.e. $x \in [0, 1]$.