

Math 8100 Assignment 2

Lebesgue measure and outer measure

Due date: Thursday the 5th of September 2019

1. Prove that if $E \subseteq \mathbb{R}$ with $m_*(E) = 0$, then $E^2 := \{x^2 \mid x \in E\}$ also has Lebesgue outer measure zero.
Hint: First consider the case when E is a bounded subset of \mathbb{R} .

[To what extent can you generalize this result?]

2. Prove that if E_1 and E_2 are measurable subsets of \mathbb{R}^n , then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

3. Suppose that $A \subseteq E \subseteq B$, where A and B are Lebesgue measurable subsets on \mathbb{R}^n .

- (a) Prove that if $m(A) = m(B) < \infty$, then E is measurable.
 (b) Give an example showing that the same conclusion does not hold if A and B have infinite measure.

4. Suppose A and B are a pair of compact subsets of \mathbb{R}^n with $A \subseteq B$, and let $a = m(A)$ and $b = m(B)$. Prove that for any c with $a < c < b$, there is a compact set E with $A \subseteq E \subseteq B$ and $m(E) = c$.

Hint: As a warm-up example, consider the one dimensional example where A a compact measurable subset of $B := [0, 1]$ and the quantity $m(A) + t - m(A \cap [0, t])$ as a function of t .

5. Let \mathcal{N} denote the non-measurable subset of $[0, 1]$ that was constructed in lecture.

- (a) Prove that if E is a measurable subset of \mathcal{N} , then $m(E) = 0$.
 (b) Show that $m_*([0, 1] \setminus \mathcal{N}) = 1$
[Hint: Argue by contradiction and pick an open set G such that $[0, 1] \setminus \mathcal{N} \subseteq G \subseteq [0, 1]$ with $m_(G) \leq 1 - \varepsilon$.]*
 (c) Conclude that there exists disjoint sets $E_1 \subseteq [0, 1]$ and $E_2 \subseteq [0, 1]$ for which

$$m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2).$$

6. (a) **The Borel-Cantelli Lemma.** Suppose $\{E_j\}_{j=1}^\infty$ is a countable family of measurable subsets of \mathbb{R}^n and that

$$\sum_{j=1}^{\infty} m(E_j) < \infty.$$

Let

$$E = \limsup_{j \rightarrow \infty} E_j := \{x \in \mathbb{R}^n : x \in E_j, \text{ for infinitely many } j\}.$$

Show that E is measurable and that $m(E) = 0$. *Hint: Write $E = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j$.*

- (b) Given any irrational x one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions a/q , with a and q relatively prime integers, such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$

However, show that the set of those $x \in \mathbb{R}$ such that there exists infinitely many fractions a/q , with a and q relatively prime integers, such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^3}$$

is a set of Lebesgue measure zero.

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that any $E \subset \mathbb{R}$ with $m_*(E) > 0$ necessarily contains a non-measurable set.
2. The **outer Jordan content** $J_*(E)$ of a set E in \mathbb{R} is defined by

$$J_*(E) = \inf \sum_{j=1}^N |I_j|,$$

where the infimum is taken over every *finite* covering $E \subseteq \cup_{j=1}^N I_j$, by intervals I_j .

- (a) Prove that $J_*(E) = J_*(\bar{E})$ for every set E (here \bar{E} denotes the closure of E).
 - (b) Exhibit a countable subset $E \subseteq [0, 1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.
3. If I is a bounded interval and $\alpha \in (0, 1)$, let us call the open interval with the same midpoint as I and length equal to α times the length of I the “open middle α th” of I . If $\{\alpha_j\}_{j=1}^\infty$ is any sequence of numbers in $(0, 1)$, then, we can define a decreasing sequence $\{K_j\}$ of closed sets as follows: $K_0 = [0, 1]$, and K_j is obtained by removing the the open middle α_j th from each of the intervals that make up K_{j-1} . The resulting limiting set $K = \bigcap_{j=1}^\infty K_j$ is called a **generalized Cantor set**.
 - (a) Suppose $\{\alpha_j\}_{j=1}^\infty$ is any sequence of numbers in $(0, 1)$.
 - i. Prove that $\prod_{j=1}^\infty (1 - \alpha_j) > 0$ if and only if $\sum_{j=1}^\infty \alpha_j < \infty$.
 - ii. Given $\beta \in (0, 1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_{j=1}^\infty (1 - \alpha_j) = \beta$.
 - (b) Given $\beta \in (0, 1)$, construct an open set G in $[0, 1]$ whose boundary has Lebesgue measure β .

Hint: Every closed nowhere dense set is the boundary of an open set.