Fourier Series Handout

Recall that for $f \in L^1(\mathbb{T})$ the *Nth partial sum of the Fourier series of f* is defined be

$$S_N f(x) = \sum_{|n|\leq N} \widehat{f}(n) e^{2\pi i n x}$$

where

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx,$$

for each $n \in \mathbb{Z}$. *Recall that* $L^1(\mathbb{T}) := \{f : \mathbb{R} \to \mathbb{C} \,:\, f(x+1) = f(x)$ *for all* $x \in \mathbb{R}$ *and* $f \in L^1([0,1])\}$.

1. (a) Prove that if $f \in L^2(\mathbb{T})$ and $\{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, then $S_N f$ converges uniformly to f for almost every x and for every x if one makes the additional assumption that $f \in C(\mathbb{T})$.

   (b) i. Prove that if $f \in C^1(\mathbb{T})$, then $S_N f$ converges uniformly to f.
   Hint: Use Cauchy-Schwarz and Parseval for f'.

   ii. Prove that if $f \in C(\mathbb{T})$ and $f' \in L^2(\mathbb{T})$, then $S_N f$ converges uniformly to f.

   (c) Prove the following strengthening of the result in part (a):

   Theorem 1 (Periodic analogue of the Fourier inversion formula)**.**
   If $f \in L^1(\mathbb{T})$ and $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$, then $S_N f \to f$ uniformly for almost every x.

   Hint: Prove that if $f \in L^1(\mathbb{T})$ and $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$, then $f \in L^2(\mathbb{T})$.

   Note that Theorem 1 (or simply the hint above) has the following consequence:

   Corollary 1. *If $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(x) = 0$ almost everywhere.*

   Both results in part (b) above in fact follow from either of the following deeper results:

   Theorem 2 (Dini's Criterion)**.** *If, for some $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, there exists $\delta > 0$ such that*

   $$\int_{|t|\leq\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty \tag{1}$$

   then $S_N f(x)$ converges to $f(x)$.

   Theorem 3 (Jordan's Criterion)**.** *If f is of bounded variation in a neighborhood of x, then $S_N f(x) \to \frac{1}{2}[f(x+) + f(x-)]$. In particular $S_N f(x) \to f(x)$ at every x at which f is continuous.*

   Note that if f is *Hölder continuous* at x, namely $|f(x+t) - f(x)| \leq C|t|^a$ for some $a > 0$, then f satisfies (1) for some $\delta > 0$. But, continuous functions need not satisfy (1) for any $\delta > 0$, in fact:

   Theorem 4 (Du Bois-Reymond)**.** *There exist $f \in C(\mathbb{T})$ whose Fourier series diverges at a point.*

2. Convergence of Fourier series is effectively a local property, and if the modifications are made outside of a neighborhood of x, then the behavior of the series at x does not change. Prove the following precise formulation of this phenomenon.

   Theorem 5 (Riemann Localization Principle)**.**
   If $f \in L^1(\mathbb{T})$ and constant on some neighborhood of a point $x \in \mathbb{T}$, then $S_N f(x) \to f(x)$.

   Hint: Re-express the Nth partial sums as

   $$S_N f(x) = f * D_N(x) := \int_0^1 f(y) D_N(x-y)\, dy$$

   where

   $$D_N(x) := \sum_{|n|\leq N} e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin \pi x} \qquad \text{(Dirichlet kernel)}.$$

3. We shall now consider the *Cesàro means* of the $S_N f$, namely

$$\sigma_N f := \frac{1}{N}\sum_{n=0}^{N-1} S_n f = f * F_N$$

where

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\Big(\frac{\sin(N\pi x)}{\sin \pi x}\Big)^2 \qquad \text{(Fejér kernel).}$$

(a) Verify that the Fejér kernel satisfies the following basic properties:

i. $0 \leq F_N(x) \leq C\dfrac{1}{N}\min\Big\{N^2, \dfrac{1}{|x|^2}\Big\}$ for some constant $C > 0$ and all $x \in [0,1]$,

ii. $\displaystyle\int_0^1 F_N(x)\,dx = 1$,

iii. $\displaystyle\lim_{N\to\infty}\int_{\delta\leq|x|\leq\frac{1}{2}} F_N(x)\,dx = 0$ for any choice of $\delta > 0$.

[Note also that $\widehat{F_N}(n) = \max\Big\{1 - \dfrac{|n|}{N}, 0\Big\}$ for all $n \in \mathbb{Z}$.]

(b) Use the *approximation to the identity*-type properties above to prove the following

Theorem 6 (Fejér's Theorem)**.** *Let* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

(i) (Classical version) *If* $f \in C(\mathbb{T})$*, then* $\sigma_N f \to f$ *uniformly on* $\mathbb{T}$ *as* $N \to \infty$.

(ii) (L^1-version) *If* $f \in L^1(\mathbb{T})$*, then* $\sigma_N f \to f$ *in* $L^1(\mathbb{T})$ *as* $N \to \infty$.

[It is also true that if $f \in L^p(\mathbb{T})$ with $1 \leq p < \infty$, then $\sigma_N f \to f$ in $L^p(\mathbb{T})$ as $N \to \infty$.]

(c) Verify that Theorem 6 gives a new proof that *Trigonometric polynomials are dense in both* $C(\mathbb{T})$ *and in* $L^1(\mathbb{T})$, and that Theorem 6 (ii) provides a new proof of Corollary 1.

4. (a) i. Prove that if f is continuous and periodic with period 1, and α is irrational, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)\,dx.$$

Hint: Use the "Periodic Weierstrass Approximation Theorem".

ii. Conclude that if α is irrational, then the sequence of fractional parts $\langle\alpha\rangle, \langle 2\alpha\rangle, \langle 3\alpha\rangle, \ldots$, where $\langle x\rangle = x - \lfloor x\rfloor$, is equidistributed in $[0,1)$, that is for every interval $(a,b) \subset [0,1)$,

$$\lim_{N\to\infty}\frac{\#\{1 \leq n \leq N : \langle n\alpha\rangle \in (a,b)\}}{N} = b - a.$$

(b) Prove that following more general criterion:

Theorem 7 (Weyl's Criterion)**.** *The following assertions concerning a given sequence* $\{\xi_n\}$ *in* $[0,1)$ *are equivalent:*

(i) *The sequence* $\{\xi_n\}$ *is equidistributed;*

(ii) *For each integer* $k \neq 0$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\xi_n} = 0;$$

(iii) *For any (Riemann) integrable function* f *on* $[0,1]$ *that is periodic with period 1*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(x)\,dx.$$