Math 8100 Exam 1

Thursday the 10th of October 2019

Answer any <u>THREE</u> of the following four problems

1. Let $m_*(E)$ denote the Lebesgue outer measure of a set $E \subseteq \mathbb{R}^n$.
   (a) Prove, using the definition of Lebesgue outer measure, that $m_*\big(\bigcup_{j=1}^{\infty} E_j\big) \leq \sum_{j=1}^{\infty} m_*(E_j)$.
   (b) Prove that for any $E \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ there exists an open set G with $E \subseteq G$ and
   $$m_*(E) \leq m_*(G) \leq m_*(E) + \varepsilon.$$
   Be sure to prove both of the inequalities above.

2. (a) Let E be a bounded subset of $\mathbb{R}^n$. Prove that the following two statements are equivalent:
   i. For any $\varepsilon > 0$, there exists an open set G and closed set F with $F \subseteq E \subseteq G$ and $m(G \setminus F) < \varepsilon$.
   ii. There exists a G_δ set V and an F_σ set H with $H \subseteq E \subseteq V$ and $m(V \setminus H) = 0$.

   (b) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of extended real-valued Lebesgue measurable functions.
   i. Prove that $\inf_k f_k$ and $\liminf\limits_{k\to\infty} f_k$ are both Lebesgue measurable functions.
   Hint: Argue that $\{x : \inf_k f_k(x) < a\} = \bigcup_k \{x : f_k(x) < a\}$ *and clearly indicate what definition/properties of Lebesgue measurable functions/sets you are using.*
   ii. Carefully state Fatou's Lemma and deduce the Monotone Convergence Theorem for from it.

3. (a) Prove that if f and g are in $L^+(\mathbb{R}^n)$, then
   $$\int (f+g) = \int f + \int g$$
   and extend this result to establish that if $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $L^+(\mathbb{R}^n)$, then
   $$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k.$$
   Hint: You may assume the linearity of the integral when it is applied to simple functions, but be sure to clearly indicate what definition/properties of the Lebesgue integral you are using.

   (b) Let $f \in L^+(\mathbb{R}^n)$ and $\mu_f(A) := \displaystyle\int_A f(x)\,dx$ for each $A \in \mathcal{M}(\mathbb{R}^n)$.
   i. Prove that if $A_1, A_2, \ldots$ are a disjoint countable collection of sets in $\mathcal{M}(\mathbb{R}^n)$, then
   $$\mu_f\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu_f(A_k).$$
   ii. Let $E_1, E_2, \ldots$ be a countable collection of sets in $\mathcal{M}(\mathbb{R}^n)$ that increase to E in the sense that $E_j \subseteq E_{j+1}$ for all j, and $E = \bigcup_{j=1}^{\infty} E_j$. Use the countable additivity of μ_f on $\mathcal{M}(\mathbb{R}^n)$ established in part (i) above to prove that
   $$\mu_f(E) = \lim_{j\to\infty} \mu_f(E_j).$$

4. (a) Prove that if $f \in L^1(\mathbb{R}^n)$, then $|f(x)| < \infty$ almost everywhere.
   (b) Prove that $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $L^1(\mathbb{R}^n)$ with $\sum\limits_{k=1}^{\infty} \|f_k\|_1 < \infty$, then $\sum\limits_{k=1}^{\infty} f_k$ in fact converges both almost everywhere and in L^1.
   Hint: You may use the result from Question 3(a) above even if you did not attempt this question.
   (c) Use the Dominated Convergence Theorem to evaluate
   $$\lim_{t\to 0} \int_0^1 \frac{e^{tx^2} - 1}{t}\, dx.$$