Math 8100 Exam 1

Wednesday the 10th of October 2018

Answer any <u>FOUR</u> of the following five problems

1. Let E be a bounded subset of $\mathbb{R}^n$. Prove that the following two statements are equivalent:
   (i) For any $\varepsilon > 0$, there exists an open set G and closed set F such that $F \subseteq E \subseteq G$ and $m(G \backslash F) < \varepsilon$.
   (ii) There exists a G_δ set V and an F_σ set H such that $H \subseteq E \subseteq V$ and $m(V \setminus H) = 0$.

2. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of extended real-valued Lebesgue measurable functions.
   (a) Prove that $\sup_k f_k$ is a Lebesgue measurable function
   (b) Deduce that if $\lim\limits_{k\to\infty} f_k(x)$ exists for every $x \in \mathbb{R}^n$ this is also a Lebesgue measurable function.

   Clearly indicate what definition/properties of Lebesgue measurable functions/sets you are using.

3. (a) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then for any $h \in \mathbb{R}$ the translated set
$$E + h := \{x + h \,:\, x \in E\}$$
   is also Lebesgue measurable and satisfies $m(E + h) = m(E)$.

   Clearly indicate what definition/properties of Lebesgue measurable sets you are using.

   (b) Prove that if f is a non-negative measurable function on $\mathbb{R}^n$ and $h \in \mathbb{R}^n$, then $\tau_h f$, defined by
$$\tau_h f(x) = f(x - h)$$
   is also a non-negative measurable function and
$$\int f(x)\,dx = \int f(x - h)\,dx.$$

   Clearly indicate what definition/properties of the Lebesgue integral you are using.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function.
   (a) Prove that
$$m\left(\{x \in \mathbb{R}^n \,:\, |f(x)| > \alpha\}\right) \leq \frac{1}{\alpha} \int |f(x)|\,dx$$
   for all $\alpha > 0$.
   (b) Prove that
$$\int |f(x)|\,dx = 0 \quad \Longleftrightarrow \quad f = 0 \;\text{ almost everywhere.}$$

5. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $L^1([0,1])$ which <u>converges in L^1</u> to a function f.
   (a) Prove that f must also be in $L^1([0,1])$.
   (b) Give an example illustrating that $\{f_k\}_{k=1}^{\infty}$ may not converges to f almost everywhere.
   (c) Prove that $\{f_k\}_{k=1}^{\infty}$ must however contain a subsequence which converges to f almost everywhere.