Among the most significant properties of Lebesgue measure are its invariance under translations and simple behavior under dilations.

<u>Theorem 1</u> Let E be a Lebesgue measurable subset of $\mathbb{R}^n$.

(i) For all $h \in \mathbb{R}^n$, the set $E+h := \{x+h : x \in E\}$ is also Lebesgue measurable, and moreover $m(E+h) = m(E)$.

(ii) For all $c \in \mathbb{R}$, the set $cE := \{cx : x \in E\}$ is also Lebesgue measurable, and moreover $m(cE) = |c|^n m(E)$.

[More generally, if T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e. $T \in GL(n, \mathbb{R})$, then the set $TE := \{Tx : x \in E\}$ is also Lebesgue measurable and $m(TE) = |\det T|\, m(E)$.]

* We could prove this now, but will do so later after having developed some integration theory *.

<u>Proof of Theorem 1:</u>

- Both results clearly hold for closed cubes and hence for <u>any</u> $E \subseteq \mathbb{R}^n$ (not necessarily measurable) we have

$$m_*(E+h) = m_*(E) \quad \underline{\text{and}} \quad m_*(cE) = |c|^n m_*(E).$$

* Check this, if you do not see it immediately *

Thus it suffices to show that the sets

$$E + h \quad \text{and} \quad cE$$

are both <u>measurable</u> (recall if E is measurable, then $m(E) := m_*(E)$)

Let $\varepsilon > 0$. Since E is measurable, we know $\exists$ open set G with

$$E \subseteq G \quad \text{and} \quad m_*(G \setminus E) \leq \varepsilon .$$

Notice that <u>$G+h$ and cG are both still open</u> and clearly

$$E+h \subseteq G+h \quad \text{and} \quad cE \subseteq cG .$$

Since

$$(G+h) \setminus (E+h) = (G \setminus E) + h$$

and $\quad cG \setminus cE = c(G \setminus E)$

it follows that

$$m_*\big((G+h) \setminus (E+h)\big) = m_*\big((G \setminus E) + h\big) = m_*(G \setminus E) \leq \varepsilon$$

<u>and</u>

$$m_*(cG \setminus cE) = m_*\big(c(G \setminus E)\big) = |c|^n m_*(G \setminus E) \leq |c|^n \varepsilon$$

these hold for <u>any</u> set

$\square$