CHAPTER

SIX

COMPLEX MEASURES

Total Variation

6.1 Introduction Let $\mathfrak{M}$ be a σ-algebra in a set X. Call a countable collection $\{E_i\}$ of members of $\mathfrak{M}$ a *partition of* E if $E_i \cap E_j = \varnothing$ whenever $i \neq j$, and if $E = \bigcup E_i$. A *complex measure* μ on $\mathfrak{M}$ is then a complex function on $\mathfrak{M}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (E \in \mathfrak{M}) \tag{1}$$

for every partition $\{E_i\}$ of E.

Observe that the convergence of the series in (1) is now part of the requirement (unlike for positive measures, where the series could either converge or diverge to ∞). Since the union of the sets E_i is not changed if the subscripts are permuted, every rearrangement of the series (1) must also converge. Hence ([26], Theorem 3.56) the series actually converges absolutely.

Let us consider the problem of finding a positive measure λ which dominates a given complex measure μ on $\mathfrak{M}$, in the sense that $|\mu(E)| \leq \lambda(E)$ for every $E \in \mathfrak{M}$, and let us try to keep λ as small as we can. Every solution to our problem (if there is one at all) must satisfy

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i) \geq \sum_{1}^{\infty} |\mu(E_i)|, \tag{2}$$

for every partition $\{E_i\}$ of any set $E \in \mathfrak{M}$, so that $\lambda(E)$ is at least equal to the supremum of the sums on the right of (2), taken over all partitions of E. This suggests that we *define a set function* $|\mu|$ *on* $\mathfrak{M}$ *by*

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \qquad (E \in \mathfrak{M}), \tag{3}$$

the supremum being taken over all partitions $\{E_i\}$ *of* E.

This notation is perhaps not the best, but it is the customary one. Note that $|\mu|(E) \geq |\mu(E)|$, but that in general $|\mu|(E)$ is not equal to $|\mu(E)|$.

It turns out, as will be proved below, that $|\mu|$ actually *is* a measure, so that our problem does have a solution. The discussion which led to (3) shows then clearly that $|\mu|$ is the minimal solution, in the sense that any other solution λ has the property $\lambda(E) \geq |\mu|(E)$ for all $E \in \mathfrak{M}$.

The set function $|\mu|$ is called the *total variation* of μ, or sometimes, to avoid misunderstanding, the *total variation measure*. The term "total variation of μ" is also frequently used to denote the number $|\mu|(X)$.

If μ is a positive measure, then of course $|\mu| = \mu$.

Besides being a measure, $|\mu|$ has another unexpected property: $|\mu|(X) < \infty$. Since $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X)$, this implies that every complex measure μ on any σ-algebra is bounded: If the range of μ lies in the complex plane, then it actually lies in some disc of finite radius. This property (proved in Theorem 6.4) is sometimes expressed by saying that μ *is of bounded variation*.

6.2 Theorem *The total variation* $|\mu|$ *of a complex measure* μ *on* $\mathfrak{M}$ *is a positive measure on* $\mathfrak{M}$.

PROOF Let $\{E_i\}$ be a partition of $E \in \mathfrak{M}$. Let t_i be real numbers such that $t_i < |\mu|(E_i)$. Then each E_i has a partition $\{A_{ij}\}$ such that

$$\sum_j |\mu(A_{ij})| > t_i \qquad (i = 1, 2, 3, \ldots). \tag{1}$$

Since $\{A_{ij}\}$ $(i, j = 1, 2, 3, \ldots)$ is a partition of E, it follows that

$$\sum_i t_i \leq \sum_{i,j} |\mu(A_{ij})| \leq |\mu|(E). \tag{2}$$

Taking the supremum of the left side of (2), over all admissible choices of $\{t_i\}$, we see that

$$\sum_i |\mu|(E_i) \leq |\mu|(E). \tag{3}$$

To prove the opposite inequality, let $\{A_j\}$ be any partition of E. Then for any fixed j, $\{A_j \cap E_i\}$ is a partition of A_j, and for any fixed i, $\{A_j \cap E_i\}$ is a partition of E_i. Hence

$$\begin{aligned}\sum_j |\mu(A_j)| &= \sum_j \left| \sum_i \mu(A_j \cap E_i) \right| \\ &\leq \sum_j \sum_i |\mu(A_j \cap E_i)| \\ &= \sum_i \sum_j |\mu(A_j \cap E_i)| \leq \sum_i |\mu|(E_i). \end{aligned} \tag{4}$$

Since (4) holds for every partition $\{A_j\}$ of E, we have

$$|\mu|(E) \leq \sum_i |\mu|(E_i). \tag{5}$$

By (3) and (5), $|\mu|$ is countably additive.

Note that the Corollary to Theorem 1.27 was used in (2) and (4).

That $|\mu|$ is not identically ∞ is a trivial consequence of Theorem 6.4 but can also be seen right now, since $|\mu|(\varnothing) = 0$. ////

6.3 Lemma *If $z_1, \ldots, z_N$ are complex numbers then there is a subset S of $\{1, \ldots, N\}$ for which*

$$\left|\sum_{k \in S} z_k\right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

PROOF Write $z_k = |z_k| e^{i\alpha_k}$. For $-\pi \leq \theta \leq \pi$, let $S(\theta)$ be the set of all k for which $\cos(\alpha_k - \theta) > 0$. Then

$$\left|\sum_{S(\theta)} z_k\right| = \left|\sum_{S(\theta)} e^{-i\theta} z_k\right| \geq \operatorname{Re} \sum_{S(\theta)} e^{-i\theta} z_k = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta).$$

Choose θ_0 so as to maximize the last sum, and put $S = S(\theta_0)$. This maximum is at least as large as the average of the sum over $[-\pi, \pi]$, and this average is $\pi^{-1} \sum |z_k|$, because

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^+(\alpha - \theta)\, d\theta = \frac{1}{\pi}$$

for every α. ////

6.4 Theorem *If μ is a complex measure on X, then*

$$|\mu|(X) < \infty.$$

PROOF Suppose first that some set $E \in \mathfrak{M}$ has $|\mu|(E) = \infty$. Put $t = \pi(1 + |\mu(E)|)$. Since $|\mu|(E) > t$, there is a partition $\{E_i\}$ of E such that

$$\sum_{i=1}^{N} |\mu(E_i)| > t$$

for some N. Apply Lemma 6.3, with $z_i = \mu(E_i)$, to conclude that there is a set $A \subset E$ (a union of some of the sets E_i) for which

$$|\mu(A)| > t/\pi > 1.$$

Setting $B = E - A$, it follows that

$$|\mu(B)| = |\mu(E) - \mu(A)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| = 1.$$

We have thus split E into disjoint sets A and B with $|\mu(A)| > 1$ and $|\mu(B)| > 1$. Evidently, at least one of $|\mu|(A)$ and $|\mu|(B)$ is ∞, by Theorem 6.2.

Now if $|\mu|(X) = \infty$, split X into A_1, B_1, as above, with $|\mu(A_1)| > 1$, $|\mu|(B_1) = \infty$. Split B_1 into A_2, B_2, with $|\mu(A_2)| > 1$, $|\mu|(B_2) = \infty$. Continuing in this way, we get a countably infinite disjoint collection $\{A_i\}$, with $|\mu(A_i)| > 1$ for each i. The countable additivity of μ implies that

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

But this series cannot converge, since $\mu(A_i)$ does not tend to 0 as $i \to \infty$. This contradiction shows that $|\mu|(X) < \infty$. ////

6.5 If μ and λ are complex measures on the same σ-algebra $\mathfrak{M}$, we define $\mu + \lambda$ and $c\mu$ by

$$\begin{aligned} (\mu + \lambda)(E) &= \mu(E) + \lambda(E) \\ (c\mu)(E) &= c\mu(E) \end{aligned} \qquad (E \in \mathfrak{M}) \tag{1}$$

for any scalar c, in the usual manner. It is then trivial to verify that $\mu + \lambda$ and $c\mu$ are complex measures. The collection of all complex measures on $\mathfrak{M}$ is thus a vector space. If we put

$$\|\mu\| = |\mu|(X), \tag{2}$$

it is easy to verify that all axioms of a normed linear space are satisfied.

6.6 Positive and Negative Variations Let us now specialize and consider a *real* measure μ on a σ-algebra $\mathfrak{M}$. (Such measures are frequently called *signed* measures.) Define $|\mu|$ as before, and define

$$\mu^+ = \tfrac{1}{2}(|\mu| + \mu), \qquad \mu^- = \tfrac{1}{2}(|\mu| - \mu). \tag{1}$$

Then both μ^+ and μ^- are positive measures on $\mathfrak{M}$, and they are bounded, by Theorem 6.4. Also,

$$\mu = \mu^+ - \mu^-, \qquad |\mu| = \mu^+ + \mu^-. \tag{2}$$

The measures μ^+ and μ^- are called the *positive* and *negative variations* of μ, respectively. This representation of μ as the difference of the positive measures μ^+ and μ^- is known as the *Jordan decomposition* of μ. Among all representations of μ as a difference of two positive measures, the Jordan decomposition has a certain minimum property which will be established as a corollary to Theorem 6.14.

Absolute Continuity

6.7 Definitions Let μ be a positive measure on a σ-algebra $\mathfrak{M}$, and let λ be an arbitrary measure on $\mathfrak{M}$; λ may be positive or complex. (Recall that a complex measure has its range in the complex plane, but that our usage of the term "positive measure" includes ∞ as an admissible value. Thus the positive measures do not form a subclass of the complex ones.)

We say that λ is *absolutely continuous* with respect to μ, and write

$$\lambda \ll \mu \tag{1}$$

if $\lambda(E) = 0$ for every $E \in \mathfrak{M}$ for which $\mu(E) = 0$.

If there is a set $A \in \mathfrak{M}$ such that $\lambda(E) = \lambda(A \cap E)$ for every $E \in \mathfrak{M}$, we say that λ is *concentrated on* A. This is equivalent to the hypothesis that $\lambda(E) = 0$ whenever $E \cap A = \varnothing$.

Suppose λ_1 and λ_2 are measures on $\mathfrak{M}$, and suppose there exists a pair of disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is concentrated on B. Then we say that λ_1 and λ_2 are *mutually singular*, and write

$$\lambda_1 \perp \lambda_2. \tag{2}$$

Here are some elementary properties of these concepts.

6.8 Proposition *Suppose, μ, λ, λ_1, and λ_2 are measures on a σ-algebra $\mathfrak{M}$, and μ is positive.*

(*a*) *If λ is concentrated on A, so is $|\lambda|$.*
(*b*) *If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.*
(*c*) *If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.*
(*d*) *If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.*
(*e*) *If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.*
(*f*) *If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.*
(*g*) *If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.*

PROOF

(*a*) If $E \cap A = \varnothing$ and $\{E_j\}$ is any partition of E, then $\lambda(E_j) = 0$ for all j. Hence $|\lambda|(E) = 0$.
(*b*) This follows immediately from (*a*).
(*c*) There are disjoint sets A_1 and B_1 such that λ_1 is concentrated on A_1 and μ on B_1, and there are disjoint sets A_2 and B_2 such that λ_2 is concentrated on A_2 and μ on B_2. Hence $\lambda_1 + \lambda_2$ is concentrated on $A = A_1 \cup A_2$, μ is concentrated on $B = B_1 \cap B_2$, and $A \cap B = \varnothing$.
(*d*) This is obvious.
(*e*) Suppose $\mu(E) = 0$, and $\{E_j\}$ is a partition of E. Then $\mu(E_j) = 0$; and since $\lambda \ll \mu$, $\lambda(E_j) = 0$ for all j, hence $\sum |\lambda(E_j)| = 0$. This implies $|\lambda|(E) = 0$.

(*f*) Since $\lambda_2 \perp \mu$, there is a set A with $\mu(A) = 0$ on which λ_2 is concentrated. Since $\lambda_1 \ll \mu$, $\lambda_1(E) = 0$ for every $E \subset A$. So λ_1 is concentrated on the complement of A.
(*g*) By (*f*), the hypothesis of (*g*) implies, that $\lambda \perp \lambda$, and this clearly forces $\lambda = 0$. ////

We come now to the principal theorem about absolute continuity. In fact, it is probably the most important theorem in measure theory. Its statement will involve σ-finite measures. The following lemma describes one of their significant properties.

6.9 Lemma *If μ is a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, then there is a function $w \in L^1(\mu)$ such that $0 < w(x) < 1$ for every $x \in X$.*

PROOF To say that μ is σ-finite means that X is the union of countably many sets $E_n \in \mathfrak{M}$ $(n = 1, 2, 3, \ldots)$ for which $\mu(E_n)$ is finite. Put $w_n(x) = 0$ if $x \in X - E_n$ and put

$$w_n(x) = 2^{-n}/(1 + \mu(E_n))$$

if $x \in E_n$. Then $w = \sum_1^\infty w_n$ has the required properties. ////

The point of the lemma is that μ can be replaced by a *finite* measure $\tilde{\mu}$ (namely, $d\tilde{\mu} = w\, d\mu$) which, because of the strict positivity of w, has *precisely* the same sets of measure 0 as μ.

6.10 The Theorem of Lebesgue-Radon-Nikodym *Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$.*

(*a*) *There is then a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that*

$$\lambda = \lambda_a + \lambda_s, \qquad \lambda_a \ll \mu, \qquad \lambda_s \perp \mu. \tag{1}$$

If λ is positive and finite, then so are λ_a and λ_s.
(*b*) *There is a unique $h \in L^1(\mu)$ such that*

$$\lambda_a(E) = \int_E h\, d\mu \tag{2}$$

for every set $E \in \mathfrak{M}$.

The pair (λ_a, λ_s) is called the *Lebesgue decomposition* of λ relative to μ. The uniqueness of the decomposition is easily seen, for if (λ_a', λ_s') is another pair which satisfies (1), then

$$\lambda_a' - \lambda_a = \lambda_s - \lambda_s', \tag{3}$$

$\lambda_a' - \lambda_a \ll \mu$, and $\lambda_s - \lambda_s' \perp \mu$, hence both sides of (3) are 0; we have used 6.8(*c*), 6.8(*d*), and 6.8(*g*).

The *existence* of the decomposition is the significant part of (a).

Assertion (b) is known as the *Radon-Nikodym* theorem. Again, uniqueness of h is immediate, from Theorem 1.39(b). Also, if h is *any* member of $L^1(\mu)$, the integral in (2) defines a measure on $\mathfrak{M}$ (Theorem 1.29) which is clearly absolutely continuous with respect to μ. The point of the Radon-Nikodym theorem is the converse: *Every* $\lambda \ll \mu$ (in which case $\lambda_a = \lambda$) is obtained in this way.

The function h which occurs in (2) is called the *Radon-Nikodym derivative* of λ_a with respect to μ. As noted after Theorem 1.29, we may express (2) in the form $d\lambda_a = h\, d\mu$, or even in the form $h = d\lambda_a/d\mu$.

The idea of the following proof, which yields both (a) and (b) at one stroke, is due to von Neumann.

PROOF Assume first that λ is a positive bounded measure on $\mathfrak{M}$. Associate w to μ as in Lemma 6.9. Then $d\varphi = d\lambda + w\, d\mu$ defines a positive bounded measure φ on $\mathfrak{M}$. The definition of the sum of two measures shows that

$$\int_X f\, d\varphi = \int_X f\, d\lambda + \int_X fw\, d\mu \tag{4}$$

for $f = \chi_E$, hence for simple f, hence for any nonnegative measurable f. If $f \in L^2(\varphi)$, the Schwarz inequality gives

$$\left|\int_X f\, d\lambda\right| \le \int_X |f|\, d\lambda \le \int_X |f|\, d\varphi \le \left\{\int_X |f|^2\, d\varphi\right\}^{1/2} \{\varphi(X)\}^{1/2}.$$

Since $\varphi(X) < \infty$, we see that

$$f \to \int_X f\, d\lambda \tag{5}$$

is a bounded linear functional on $L^2(\varphi)$. We know that every bounded linear functional on a Hilbert space H is given by an inner product with an element of H. Hence there exists a $g \in L^2(\varphi)$ such that

$$\int_X f\, d\lambda = \int_X fg\, d\varphi \tag{6}$$

for every $f \in L^2(\varphi)$.

Observe how the completeness of $L^2(\varphi)$ was used to guarantee the existence of g. Observe also that although g is defined uniquely as an element of $L^2(\varphi)$, g is determined only a.e. $[\varphi]$ as a point function on X.

Put $f = \chi_E$ in (6), for any $E \in \mathfrak{M}$ with $\varphi(E) > 0$. The left side of (6) is then $\lambda(E)$, and since $0 \le \lambda \le \varphi$, we have

$$0 \le \frac{1}{\varphi(E)} \int_E g\, d\varphi = \frac{\lambda(E)}{\varphi(E)} \le 1. \tag{7}$$

Hence $g(x) \in [0, 1]$ for almost all x (with respect to φ), by Theorem 1.40. We may therefore assume that $0 \le g(x) \le 1$ for every $x \in X$, without affecting (6), and we rewrite (6) in the form

$$\int_X (1 - g) f\, d\lambda = \int_X fgw\, d\mu. \tag{8}$$

Put

$$A = \{x: 0 \le g(x) < 1\}, \qquad B = \{x: g(x) = 1\}, \tag{9}$$

and define measures λ_a and λ_s by

$$\lambda_a(E) = \lambda(A \cap E), \qquad \lambda_s(E) = \lambda(B \cap E), \tag{10}$$

for all $E \in \mathfrak{M}$.

If $f = \chi_B$ in (8), the left side is 0, the right side is $\int_B w\, d\mu$. Since $w(x) > 0$ for all x, we conclude that $\mu(B) = 0$. Thus $\lambda_s \perp \mu$.

Since g is bounded, (8) holds if f is replaced by

$$(1 + g + \cdots + g^n)\chi_E$$

for $n = 1, 2, 3, \ldots$, $E \in \mathfrak{M}$. For such f, (8) becomes

$$\int_E (1 - g^{n+1})\, d\lambda = \int_E g(1 + g + \cdots + g^n) w\, d\mu. \tag{11}$$

At every point of B, $g(x) = 1$, hence $1 - g^{n+1}(x) = 0$. At every point of A, $g^{n+1}(x) \to 0$ monotonically. The left side of (11) converges therefore to $\lambda(A \cap E) = \lambda_a(E)$ as $n \to \infty$.

The integrands on the right side of (11) increase monotonically to a nonnegative measurable limit h, and the monotone convergence theorem shows that the right side of (11) tends to $\int_E h\, d\mu$ as $n \to \infty$.

We have thus proved that (2) holds for every $E \in \mathfrak{M}$. Taking $E = X$, we see that $h \in L^1(\mu)$, since $\lambda_a(X) < \infty$.

Finally, (2) shows that $\lambda_a \ll \mu$, and the proof is complete for positive λ.

If λ is a complex measure on $\mathfrak{M}$, then $\lambda = \lambda_1 + i\lambda_2$, with λ_1 and λ_2 real, and we can apply the preceding case to the positive and negative variations of λ_1 and λ_2. ////

If both μ and λ are positive and σ-finite, most of Theorem 6.10 is still true. We can now write $X = \bigcup X_n$, where $\mu(X_n) < \infty$ and $\lambda(X_n) < \infty$, for $n = 1, 2, 3, \ldots$. The Lebesgue decompositions of the measures $\lambda(E \cap X_n)$ still give us a Lebesgue decomposition of λ, and we still get a function h which satisfies Eq. 6.10(2); however, it is no longer true that $h \in L^1(\mu)$, although h is "locally in L^1," i.e., $\int_{X_n} h\, d\mu < \infty$ for each n.

Finally, if we go beyond σ-finiteness, we meet situations where the two theorems under consideration actually fail. For example, let μ be Lebesgue measure on (0, 1), and let λ be the counting measure on the σ-algebra of all Lebesgue measurable sets in $(0, 1)$. *Then λ has no Lebesgue decomposition relative to μ, and although $\mu \ll \lambda$ and μ is bounded, there is no $h \in L^1(\lambda)$ such that $d\mu = h\,d\lambda$.* We omit the easy proof.

The following theorem may explain why the word "continuity" is used in connection with the relation $\lambda \ll \mu$.

6.11 Theorem *Suppose μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive, and λ is complex. Then the following two conditions are equivalent:*

(*a*) $\lambda \ll \mu$.

(*b*) *To every $\epsilon > 0$ corresponds a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$ with $\mu(E) < \delta$.*

Property (*b*) is sometimes used as the definition of absolute continuity. However, (*a*) does not imply (*b*) if λ is a positive unbounded measure. For instance, let μ be Lebesgue measure on $(0, 1)$, and put

$$\lambda(E) = \int_E t^{-1}\,dt$$

for every Lebesgue measurable set $E \subset (0, 1)$.

PROOF Suppose (*b*) holds. If $\mu(E) = 0$, then $\mu(E) < \delta$ for every $\delta > 0$, hence $|\lambda(E)| < \epsilon$ for every $\epsilon > 0$, so $\lambda(E) = 0$. Thus (*b*) implies (*a*).

Suppose (*b*) is false. Then there exists an $\epsilon > 0$ and there exist sets $E_n \in \mathfrak{M}$ $(n = 1, 2, 3, \ldots)$ such that $\mu(E_n) < 2^{-n}$ but $|\lambda(E_n)| \geq \epsilon$. Hence $|\lambda|(E_n) \geq \epsilon$. Put

$$A_n = \bigcup_{i=n}^{\infty} E_i, \qquad A = \bigcap_{n=1}^{\infty} A_n. \tag{1}$$

Then $\mu(A_n) < 2^{-n+1}$, $A_n \supset A_{n+1}$, and so Theorem 1.19(*e*) shows that $\mu(A) = 0$ and that

$$|\lambda|(A) = \lim_{n \to \infty} |\lambda|(A_n) \geq \epsilon > 0,$$

since $|\lambda|(A_n) \geq |\lambda|(E_n)$.

It follows that we do *not* have $|\lambda| \ll \mu$, hence (*a*) is false, by Proposition 6.8(*e*). ////

Consequences of the Radon-Nikodym Theorem

6.12 Theorem *Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that*

$$d\mu = h\,d|\mu|. \tag{1}$$

By analogy with the representation of a complex number as the product of its absolute value and a number of absolute value 1, Eq. (1) is sometimes referred to as the *polar representation* (or *polar decomposition*) of μ.

PROOF It is trivial that $\mu \ll |\mu|$, and therefore the Radon-Nikodym theorem guarantees the existence of some $h \in L^1(|\mu|)$ which satisfies (1).

Let $A_r = \{x : |h(x)| < r\}$, where r is some positive number, and let $\{E_j\}$ be a partition of A_r. Then

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h\,d|\mu| \right| \leq \sum_j r|\mu|(E_j) = r|\mu|(A_r),$$

so that $|\mu|(A_r) \leq r|\mu|(A_r)$. If $r < 1$, this forces $|\mu|(A_r) = 0$. Thus $|h| \geq 1$ a.e.

On the other hand, if $|\mu|(E) > 0$, (1) shows that

$$\left| \frac{1}{|\mu|(E)} \int_E h\,d|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1.$$

We now apply Theorem 1.40 (with the closed unit disc in place of S) and conclude that $|h| \leq 1$ a.e.

Let $B = \{x \in X : |h(x)| \neq 1\}$. We have shown that $|\mu|(B) = 0$, and if we redefine h on B so that $h(x) = 1$ on B, we obtain a function with the desired properties. ////

6.13 Theorem *Suppose μ is a positive measure on $\mathfrak{M}$, $g \in L^1(\mu)$, and*

$$\lambda(E) = \int_E g\,d\mu \qquad (E \in \mathfrak{M}). \tag{1}$$

Then

$$|\lambda|(E) = \int_E |g|\,d\mu \qquad (E \in \mathfrak{M}). \tag{2}$$

PROOF By Theorem 6.12, there is a function h, of absolute value 1, such that $d\lambda = h\,d|\lambda|$. By hypothesis, $d\lambda = g\,d\mu$. Hence

$$h\,d|\lambda| = g\,d\mu.$$

This gives $d|\lambda| = \bar{h}g\,d\mu$. (Compare with Theorem 1.29.)

Since $|\lambda| \geq 0$ and $\mu \geq 0$, it follows that $\bar{h}g \geq 0$ a.e. $[\mu]$, so that $\bar{h}g = |g|$ a.e. $[\mu]$. ////

6.14 The Hahn Decomposition Theorem *Let μ be a real measure on a σ-algebra $\mathfrak{M}$ in a set X. Then there exist sets A and $B \in \mathfrak{M}$ such that*

$A \cup B = X$, $A \cap B = \varnothing$, and such that the positive and negative variations μ^+ and μ^- of μ satisfy

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = -\mu(B \cap E) \qquad (E \in \mathfrak{M}). \tag{1}$$

In other words, X is the union of two disjoint measurablé sets A and B, such that "A carries all the positive mass of μ" [since (1) implies that $\mu(E) \geq 0$ if $E \subset A$] and "B carries all the negative mass of μ" [since $\mu(E) \leq 0$ if $E \subset B$]. The pair (A, B) is called a Hahn decomposition of X, induced by μ.

PROOF By Theorem 6.12, $d\mu = h\, d|\mu|$, where $|h| = 1$. Since μ is real, it follows that h is real (a.e., and therefore everywhere, by redefining on a set of measure 0), hence $h = \pm 1$. Put

$$A = \{x: h(x) = 1\}, \qquad B = \{x: h(x) = -1\}. \tag{2}$$

Since $\mu^+ = \frac{1}{2}(|\mu| + \mu)$, and since

$$\tfrac{1}{2}(1 + h) = \begin{cases} h & \text{on } A, \\ 0 & \text{on } B, \end{cases} \tag{3}$$

we have, for any $E \in \mathfrak{M}$,

$$\mu^+(E) = \frac{1}{2}\int_E (1 + h)\, d|\mu| = \int_{E \cap A} h\, d|\mu| = \mu(E \cap A). \tag{4}$$

Since $\mu(E) = \mu(E \cap A) + \mu(E \cap B)$ and since $\mu = \mu^+ - \mu^-$, the second half of (1) follows from the first. ////

Corollary *If $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.*

This is the minimum property of the Jordan decomposition which was mentioned in Sec. 6.6.

PROOF Since $\mu \leq \lambda_1$, we have

$$\mu^+(E) = \mu(E \cap A) \leq \lambda_1(E \cap A) \leq \lambda_1(E). \qquad ////$$