Math 8100 Practice Exam 2

November 2014

This is much longer than an actual exam would be, but should hopefully still help with your studying!

1. **Fubini-Tonelli**

   (a) Carefully state Tonelli's theorem for a non-negative function $F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

   (b) Let $f : \mathbb{R}^n \to [0,\infty]$ and
   $$\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,:\, 0 \leq t \leq f(x)\}.$$
   Prove the validity of the following two statements:

   i. f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

   ii. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then
   $$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty m(\{x \in \mathbb{R}^n \,:\, f(x) \geq t\})\,dt$$

   Hint: In any part of this question you may use, without proof, the fact that if $f(x)$ is a Lebesgue measurable function on $\mathbb{R}^n$, then $F(x,t) = f(x)$ is a Lebesgue measurable function on $\mathbb{R}^n \times \mathbb{R}$.

2. **Convolutions and the Fourier transform**

   (a) i. Give a definition for the *convolution* $f * g$ of two integrable functions f and g on $\mathbb{R}^n$.

   ii. Prove that if f and g are bounded integrable functions on $\mathbb{R}^n$, then
   $$\lim_{|x|\to\infty} f * g(x) = 0$$

   Hint: Recall that $|x| \leq |x-y| + |y|$

   * Can you find an example of two non-negative integrable functions f and g for which
   $$\lim_{|x|\to\infty} f * g(x) \neq 0?$$

   (b) Let $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ and $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$. Prove that if $f \in L^1(\mathbb{R}^n)$, then
   $$\lim_{t\to 0} \int \big| f * \varphi_t(x) - f(x) \big|\,dx = 0.$$

   (c) i. Give a definition of the *Fourier transform* of an integrable function f on $\mathbb{R}^n$.

   ii. Give an outline of the proof of the *Fourier inversion formula*, namely that if f <u>and</u> $\widehat{f}$ are functions in $L^1(\mathbb{R}^n)$, then
   $$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x\cdot\xi}\,d\xi$$
   for almost every $x \in \mathbb{R}^n$.

   iii. Give an example, no proofs required, of a function in $L^1(\mathbb{R}^n)$ whose Fourier transform is not in $L^1(\mathbb{R}^n)$.

3. **Hilbert spaces**

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal set in a Hilbert space H.

(a) Let x be any element of H. Verify that

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

for any $N \in \mathbb{N}$ and deduce from this *Bessel's inequality*, namely that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

(b) Let $\{a_n\}$ be any element of $\ell^2(\mathbb{N})$. Prove that there exist $x \in H$ such that $a_n = \langle x, u_n \rangle$ for all $n \in \mathbb{N}$, and moreover that x may be chosen so that

$$\|x\| = \Big(\sum_{n=1}^{\infty} |a_n|^2 \Big)^{1/2}.$$

(c) Prove that if $\{u_n\}_{n=1}^{\infty}$ is *complete*, namely that it has the property that $x = 0$ whenever $\langle x, u_n \rangle = 0$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$$

for every $x \in H$.

4. L^p **spaces**

(a) (Hölder's inequality) Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 < p, q < \infty$ conjugate exponents. Prove that $fg \in L^1(\mathbb{R}^n)$ with

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Hint: You may use, without proof, the fact that $AB \leq A^p/p + B^q/q$ for any non-negative real numbers A and B.

(b) (Minkowski's inequality) Let $f, g \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. Prove that $f + g \in L^p(\mathbb{R}^n)$ with

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

(c) Let $X = [0, 1]$.

i. Give a definition for the Banach space, $L^\infty(X)$, of essentially bounded functions on X.

ii. Let f be a non-negative measurable function on X. Prove that $\lim_{p \to \infty} \int_X f(x)^p \, dx$ is either ∞ or $m(\{f^{-1}(1)\})$, and characterize the collection of functions f of each type.

5. **Dual spaces**

Let $(X, \|\cdot\|)$ be a normed vector space.

(a) Give the definition of what it means for a mapping $L : X \to \mathbb{C}$ to be a *linear functional.*

(b) Give the definition of what it means for a *linear functional* L to be *bounded* and prove that this notion is equivalent to L being continuous.

(c) Prove that the dual space of X, namely the space of all continuous linear functionals on X, denoted by X^* is *Banach space* when equipped with the *operator norm.*