Math 8100 Assignment 9
Basic Function Spaces

Due date: Friday the 14th of November 2014

1. Prove the following basic properties of $L^\infty = L^\infty(X)$, where X is a measurable subset of $\mathbb{R}^n$:

   (a) $\|\cdot\|_\infty$ is a norm on L^∞ and when equipped with this norm L^∞ is a Banach space.

   (b) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathbb{R}^n$ such that $m(E^c) = 0$ and $f_n \to f$ uniformly on E.

   (c) Simple functions are dense in L^∞, but continuous functions with compact support are not.

   Recall that if $X \subseteq \mathbb{R}^n$ is measurable and f is a measurable function on X, then we define

   $$\|f\|_\infty = \inf\{a \geq 0 \, : m(\{x \in X \, : \, |f(x)| > a\}) = 0\},$$

   with the convention that $\inf \emptyset = \infty$*, and*

   $$L^\infty = L^\infty(X) = \{f : X \to \mathbb{C} \text{ measuarable} \, : \, \|f\|_\infty < \infty\},$$

   with the usual convention that two functions that are equal a.e. define the same element of L^∞. Thus $f \in L^\infty$ if and only if there is a bounded function g such that $f = g$ almost everywhere; we can take $g = f\chi_E$ where $E = \{x \, : \, |f(x)| \leq \|f\|_\infty\}$.

2. Let $C([0,1])$ denote the space of all continuous $\mathbb{C}$-valued functions on $[0,1]$.

   (a) Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \max_{x\in[0,1]} |f(x)|$.

   (b) Prove that $C([0,1])$ is <u>not</u> complete under either the $L^1([0,1])$ or $L^2([0,1])$ norms.

3. Let $X \subseteq \mathbb{R}^n$ be measurable.

   (a) i. Prove that if $m(X) < \infty$, then

   $$L^\infty(X) \subset L^2(X) \subset L^1(X) \tag{1}$$

   with strict inclusion in each case, and that for any measurable $f : X \to \mathbb{C}$ one in fact has

   $$\|f\|_{L^1(X)} \leq m(X)^{1/2}\|f\|_{L^2(X)} \leq m(X)\|f\|_{L^\infty(X)}.$$

   ii. Give examples to show that no such result of the form (1) can hold if one drops the assumption that $m(x) < \infty$.

   (b) Prove that

   $$\underbrace{L^1(X) \cap L^\infty(X) \subset L^2(X)}_{(\star)} \subset L^1(X) + L^\infty(X)$$

   and that in addition to $(\star)$ one in fact has

   $$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2}\|f\|_{L^\infty(X)}^{1/2}$$

   for any measurable function $f : X \to \mathbb{C}$.

4. Prove that

   $$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$$

   with strict inclusion in each case, and that for any sequence $a = \{a_j\}_{j\in\mathbb{Z}}$ of complex numbers one in fact has

   $$\|a\|_{\ell^\infty(\mathbb{Z})} \leq \|a\|_{\ell^2(\mathbb{Z})} \leq \|a\|_{\ell^1(\mathbb{Z})}.$$

Recall that for $p = 1, 2, \infty$ we define

$$\ell^p(\mathbb{Z}) = \{a = \{a_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{C} : \|a\|_{\ell^p(\mathbb{Z})} < \infty\}$$

where

$$\|a\|_{\ell^1(\mathbb{Z})} = \sum_{j=-\infty}^{\infty} |a_j|, \quad \|a\|_{\ell^2(\mathbb{Z})} = \Big(\sum_{j=-\infty}^{\infty} |a_j|^2\Big)^{1/2}, \text{ and } \|a\|_{\ell^\infty(\mathbb{Z})} = \sup_j |a_j|.$$

5. Let H be a Hilbert space with orthonormal basis $\{u_n\}_{n=1}^{\infty}$.
   (a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Prove that

   $$\sum_{n=1}^{\infty} a_n u_n \text{ converges in } H \iff \sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

   and moreover that if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, then $\Big\|\sum_{n=1}^{\infty} a_n u_n\Big\| = \Big(\sum_{n=1}^{\infty} |a_n|^2\Big)^{1/2}$.
   (b) i. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1}$ for all $n \in \mathbb{N}$? If L exists, find its norm.
   ii. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1/2}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

6. Prove that if $f \in L^2(\mathbb{T}) := \{f \in L^2([0,1]) : f(0) = f(1)\}$ and $\{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, where

   $$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\, dx,$$

   then the *Nth partial sum of the Fourier series of f*, namely

   $$S_N f(x) = \sum_{|n|\le N} \widehat{f}(n)e^{2\pi inx}$$

   converges *uniformly* to $f(x)$ as $N \to \infty$ for almost every $x \in [0,1]$ and for every $x \in [0,1]$ if one makes the additional assumption that $f \in C(\mathbb{T})$, namely 1-periodic and continuous.

   Recall that the sequence of functions $\{e_n\}_{n\in\mathbb{Z}}$, defined for each $n \in \mathbb{Z}$ and $x \in [0,1]$ by $e_n(x) = e^{2\pi inx}$, forms an orthonormal basis for the Hilbert space $L^2([0,1])$ equipped with its usual inner product

   $$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\, dx$$

   and that (for each $n \in \mathbb{Z}$) the nth Fourier coefficient of f is defined to be $\widehat{f}(n) := \langle f, e_n\rangle$.

Extra Challenge Problems

Not to be handed in with the assignment

1. (a) Prove that if $f \in C^1(\mathbb{T})$, then $S_N f$ converges uniformly to f.
   (b) Prove that if $f \in C(\mathbb{T})$ and $f' \in L^2(\mathbb{T})$, then $S_N f$ converges uniformly to f.

   Both of these results in fact follow from the following deeper result:

   Theorem 1 (Dini's Criterion). *If, for some $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, there exists $\delta > 0$ such that*

   $$\int_{|t|\le\delta} \Big|\frac{f(x+t) - f(x)}{t}\Big|\, dt < \infty \tag{2}$$

   then $S_N f(x)$ converges to $f(x)$.

   Note that if f is *Hölder continuous* at x, namely $|f(x+t) - f(x)| \le C|t|^a$ for some $a > 0$, then f satisfies (2) for some $\delta > 0$. But, continuous functions need not satisfy (2) for any $\delta > 0$, in fact:

   Theorem 2 (Du Bois-Reymond). *There exist $f \in C(\mathbb{T})$ whose Fourier series diverges at a point.*