Math 8100 Assignment 7
The Fourier Transform

Due date: Friday the 24th of October 2014

Recall that we have defined the Fourier transform of an integrable function f on $\mathbb{R}^n$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$$

where $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$ and the convolution of two integrable functions f and g on $\mathbb{R}^n$ by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

1. (a) Prove that if $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f*g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for all $\xi \in \mathbb{R}^n$.
   (b) Conclude from part (a) that
      i. if $f, g, h \in L^1(\mathbb{R}^n)$, then $f*g = g*f$ and $(f*g)*h = f*(g*h)$ almost everywhere.
      ii. there does not exist $I \in L^1(\mathbb{R}^n)$ such that $f*I = f$ almost everywhere for all $f \in L^1(\mathbb{R}^n)$.

2. Let $f \in L^1(\mathbb{R}^n)$.
   (a) Show that if $y \in \mathbb{R}^n$ and
      i. $g(x) = f(x-y)$ for all $x \in \mathbb{R}^n$, then $\widehat{g}(\xi) = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$.
      ii. $h(x) = e^{2\pi i x\cdot y}f(x)$ for all $x \in \mathbb{R}^n$, then $\widehat{h}(\xi) = \widehat{f}(\xi - y)$ for all $\xi \in \mathbb{R}^n$.
   (b) Show that if T be a non-singular linear transformation of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ denote its inverse transpose, then
   $$\widehat{f\circ T}(\xi) = \frac{1}{|\det T|}\widehat{f}(S\xi)$$
   for all $\xi \in \mathbb{R}^n$.

3. (a) Let $f \in L^1(\mathbb{R})$.
      i. Let $g(x) = xf(x)$. Show that if $g \in L^1$, then $\widehat{f}$ is differentiable and $\frac{d}{d\xi}\widehat{f}(\xi) = -2\pi i\,\widehat{g}(\xi)$.
      ii. Let $f \in C_0^1(\mathbb{R})$ and $h(x) = \frac{d}{dx}f(x)$. Show that if $h \in L^1$, then $\widehat{h}(\xi) = 2\pi i\xi\widehat{f}(\xi)$.
      Recall that $C_0^1(\mathbb{R})$ is the collection of functions in $C^1(\mathbb{R})$ which vanishes at infinity.
   (b) Let $G(x) = e^{-\pi x^2}$. By considering the derivative of $\widehat{G}(\xi)/G(\xi)$, show that $\widehat{G}(\xi) = G(\xi)$.
   Hint: You may also want to use the fact that $\int_{\mathbb{R}} G(x)\,dx = 1$ (see "challenge" problem).

4. The functions D, F, and P defined below are all bounded $L^+(\mathbb{R})$ functions with integrals equal to 1.
   (a) Show that if
   $$D(x) = \begin{cases} 1 & \text{if } |x| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$
   then
   $$\widehat{D}(\xi) = \frac{\sin \pi\xi}{\pi\xi}.$$
   Hint: Note that this gives an explicit example of a function which is not in $L^1(\mathbb{R})$, but yet is the Fourier transform of an L^1 function. See Question 5 for additional higher dimensional examples.

(b) Let

$$F(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

i. Show that

$$\widehat{F}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi} \right)^2.$$

Hint: It may help to write $\widehat{F}(\xi) = h(\xi) + h(-\xi)$ *where* $h(\xi) = e^{2\pi i\xi} \int_0^1 y e^{-2\pi i y\xi} dy$.

ii. Find the Fourier transform of the function

$$f(x) = \left(\frac{\sin \pi x}{\pi x} \right)^2.$$

Be careful to fully justify your answer.

(c) Show that if

$$P(x) = \frac{1}{\pi} \frac{1}{1+x^2}.$$

then

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|} e^{2\pi i x\xi}\, d\xi = P(x)$$

and hence that

$$\widehat{P}(\xi) = e^{-2\pi|\xi|}.$$

Be careful to fully justify your answer.

Remark: *In Questions 3b and 4 above D is for Dirichlet, F is for Fejér, P is for Poisson, and G is for Gauss-Weierstrass. The respective "approximate identities", namely* $\{D_t\}_{t>0}$, $\{F_{1/t}\}_{t>0}$, $\{P_t\}_{t>0}$, *and* $\{G_{\sqrt{t}}\}_{t>0}$, *are generally referred to as Dirichlet, Fejér, Poisson, and Gauss-Weierstrass kernels.*

5. Show that for any $\varepsilon > 0$ the function $F(\xi) = (1+|\xi|^2)^{-\varepsilon}$ is the Fourier transform of an $L^1(\mathbb{R}^n)$ function.
Hint: Consider the function

$$f(x) = \int_0^{\infty} G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1}\, dt,$$

where $G_t(x) = t^{-n} e^{-\pi|x|^2/t^2}$. *Now use Fubini/Tonelli to prove that* $f \in L^1(\mathbb{R}^n)$ *with* $\widehat{f}(\xi) = F(\xi)\|f\|_1$.

Extra Challenge Problems

Not to be handed in with the assignment

1. By considering the iterated integral

$$\int_0^{\infty} \left(\int_0^{\infty} x e^{-x^2(1+y^2)}\, dx \right) dy$$

show (with justification) that

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

and hence that

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1.$$