Math 8100 Assignment 6
Repeated Integration

Due date: Wednesday the 15th of October 2014

1. Let $f \in L^1([0,1])$, and for each $x \in [0,1]$ define
$$g(x) = \int_x^1 \frac{f(t)}{t}\, dt.$$
Show that $g \in L^1([0,1])$ and that
$$\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx.$$

2. Suppose that $f \in L^1(\mathbb{R}^n)$. Show that
$$\int_{\mathbb{R}^n} |f(x)|\, dx = \int_0^\infty m(\{x \in \mathbb{R}^n \,:\, |f(x)| > t\})\, dt.$$

3. Let $A, B \subseteq \mathbb{R}^n$ be bounded measurable sets with positive Lebesgue measure. For each $t \in \mathbb{R}^n$ define the function
$$g(t) = m\left(A \cap (t - B)\right)$$
where $t - B = \{t - b \,:\, b \in B\}$.
   (a) Prove that g is a continuous function and
   $$\int_{\mathbb{R}^n} g(t)\, dt = m(A)\, m(B).$$
   (b) Conclude that the sumset
   $$A + B = \{a + b \,:\, a \in A \text{ and } b \in B\}$$
   contains a non-empty open subset of $\mathbb{R}^n$.

4. Let $f \in L^1(\mathbb{R})$. For any $h > 0$ we define
$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy$$
   (a) Prove that for all $h > 0$,
   $$\int_{\mathbb{R}} |A_h(f)(x)|\, dx \leq \int_{\mathbb{R}} |f(x)|\, dx.$$
   (b) Prove that
   $$\lim_{h \to 0^+} \int_{\mathbb{R}} |A_h(f)(x) - f(x)|\, dx = 0.$$

5. (a) Prove that
   $$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \infty.$$
   (b) By considering the iterated integral
   $$\int_0^\infty \left(\int_0^\infty x e^{-xy} (1 - \cos y)\, dy \right) dx$$
   show (with justification) that
   $$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. (a) Prove that if $A, B \in \mathcal{M}(\mathbb{R})$, then $A \times B \in \mathcal{M}(\mathbb{R}^2)$ with $m(A \times B) = m(A)m(B)$.

   (b) i. The *continuum hypothesis* asserts that whenever S is an infinite subset of $\mathbb{R}$, then either S is countable, or S has the cardinality of $\mathbb{R}$. Accepting the validity of the continuum hypothesis show that there exists an ordering $\prec$ of $\mathbb{R}$ with the property that for each $y \in \mathbb{R}$ the set $\{x \in \mathbb{R} : x \prec y\}$ is at most countable.

   ii. Given the ordering $\prec$ from part (i) we define

$$E = \{(x, y) \in [0,1] \times [0,1] : x \prec y\}.$$

   Show that E is not measurable, even though the slices

$$E_x = \{y \in \mathbb{R} : (x, y) \in E\} \quad \text{and} \quad E^y = \{x \in \mathbb{R} : (x, y) \in E\}$$

   are both measurable with $m(E_x) = 1$ and $m(E^y) = 0$ for each $x, y \in [0,1]$.

   Hint for part (i): Let $\prec$ denote a well-ordering of $\mathbb{R}$, and define

$$X = \{y \in \mathbb{R} : \textit{the set } \{x : x \prec y\} \textit{ is not countable}\}.$$

   If X is empty we are done. Otherwise, consider the smallest element y' in X, and use the continuum hypothesis.