Math 8100 Assignment 5
Lebesgue Integration II

Due date: Friday the 26th of September 2014

1. (a) Give an example of a continuous function f in $L^1(\mathbb{R})$ for which $f(x) \nrightarrow 0$ as $|x| \to \infty$.
   (b) Prove that if $f \in L^1(\mathbb{R})$ and uniformly continuous, then $\lim_{|x|\to\infty} f(x) = 0$.

2. Prove that if $\int_E f = 0$ for all Lebesgue measurable subsets E of $\mathbb{R}^n$, then $f = 0$ almost everywhere.

3. Prove that if $f \in L^1(\mathbb{R})$, then $F(x) := \int_{-\infty}^{x} f(t)\,dt$ defines a uniformly continuous function on $\mathbb{R}$.

4. Recall that for a given $f \in L^1$, the *Fourier transform of f* is defined by
$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx.$$
   Prove the so-called *Riemann-Lebesgue lemma*, namely that
$$\widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$
   Hint: Write $\widehat{f}(\xi) = \frac{1}{2}\int [f(x) - f(x-\xi')]e^{-2\pi i x\cdot\xi}\,dx$, *where* $\xi' = \frac{\xi}{2|\xi|^2}$.

5. A sequence $\{f_k\}$ of integrable functions on $\mathbb{R}^n$ is said to *converge in measure* to f if for every $\varepsilon > 0$,
$$\lim_{k\to\infty} m(\{x \in \mathbb{R}^n : |f_k(x) - f(x)| \geq \varepsilon\}) = 0.$$
   (a) Prove that if $f_k \to f$ in L^1 then $f_k \to f$ in measure.
   (b) Give an example to show that the converse of Question 5a is false.
   (c) Prove that if we make the additional assumption that there exists an integrable function g such that $|f_k| \leq g$ for all k, then $f_k \to f$ in measure implies that
      i. * (Bonus points) $f \in L^1$
         Hint: First show that $\{f_k\}$ contains a subsequence which converges to f almost everywhere.
      ii. $f_k \to f$ in L^1.
         Hint: Try using absolute continuity and "small tails property" of the Lebesgue integral.

6. (a) (A Generalized Dominated Convergence Theorem) Let $\{g_n\}, \{h_n\} \subseteq L^1$ and $g, h \in L^1$ with $g_k \to g$ and $h_k \to h$ almost everywhere. Prove that if $|g_k| \leq h_k$ for all k and $\int h_k \to \int h$, then $\int g_k \to \int g$.
   Hint: Applying Fatou's lemma to something like $h_k - |g_k|$ is an option, but not the only one.
   (b) Suppose $\{f_k\} \subseteq L^1$ and $f \in L^1$ and $f_k \to f$ almost everywhere. Prove that
$$\int |f - f_k| \to 0 \quad \Longleftrightarrow \quad \int |f_k| \to \int |f|.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Let $\Omega \subseteq \mathbb{R}^n$ be measurable with $m(\Omega) < \infty$. A set $\Phi \subseteq L^1(\Omega)$ is said to be *uniformly integrable* if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f \in \Phi$ and $E \subseteq \Omega$ is measurable with $m(E) < \delta$, then
$$\int_E |f(x)|\,dx < \varepsilon.$$
   (a) Prove that if $f \in L^1(\Omega)$ and $\{f_k\}$ is a uniformly integrable sequence of functions in $L^1(\Omega)$ such that $f_k \to f$ almost everywhere on Ω, then $f_k \to f$ in $L^1(\Omega)$.
   (b) Is it necessary to assume that $f \in L^1(\Omega)$?