Math 8100 Assignment 4
Lebesgue Integration

Due date: Friday the 19th of September 2014

Definition. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$.

We say that a measurable function $f : E \to \mathbb{C}$ is *integrable on* E if $\displaystyle\int_E |f(x)|\,dx < \infty$.

1. Let f be an integrable function on $\mathbb{R}^n$.

   (a) Prove that $\{x : |f(x)| = \infty\}$ has measure equal to zero.

   (b) Let $\varepsilon > 0$. Prove that there exists a measurable set E with $m(E) < \infty$ for which

$$\int_E |f| > \left(\int |f|\right) - \varepsilon.$$

2. Suppose $f \geq 0$, and let $E_{2^k} = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then $\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\}$, and the sets F_k are disjoint. Prove that

$$\int |f(x)| < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

3. Prove the following:

   (a)

$$\int_{\{x\in\mathbb{R}^n : |x|\leq 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p < n.$$

   (b)

$$\int_{\{x\in\mathbb{R}^n : |x|\geq 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p > n.$$

   Hint: One possible approach is to use the first equivalence in Question 2 above. I suggest however that in this case you also try simply writing $\mathbb{R}^n$ as a disjoint union of the annuli $A_k = \{2^k < |x| \leq 2^{k+1}\}$.

4. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ a.e. in $\mathbb{R}$,

$$\lim_{n\to\infty} \int |f_n(x)|\,dx = A \qquad \text{and} \qquad \int |g(x)|\,dx = B.$$

   (a) Prove that

$$\lim_{n\to\infty} \int |f_n(x) - g(x)|\,dx = A - B.$$

   (b) Give an example of a sequence $\{f_n\}$ of such functions for which $A \neq B$.

5. Given any integrable function f on $\mathbb{R}^n$, the *Fourier transform of* f is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\,dx$$

   where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. Show that $\widehat{f}$ is a bounded continuous function of ξ.

6. (a) Suppose that $f(x)$ and $xf(x)$ are both integrable functions on $\mathbb{R}$. Prove that the function

$$F(t) = \int_{\mathbb{R}} f(x) \cos(tx)\, dx.$$

is differentiable at every t and find a formula for $F'(t)$.

(b) Giving complete justification, evaluate

$$\lim_{t \to 0} \int_0^1 \frac{e^{t\sqrt{x}} - 1}{t}\, dx.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Assume Fatou's theorem and deduce the monotone convergence theorem from it.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and $f : E \times [a,b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be such that for each $t \in [a,b]$, $f(x,t)$ is an integrable function of x. Let $F(t) = \int f(x,t)\, dx$.

   (a) Suppose that there exists an integrable function g such that $|f(x,t)| \leq g(x)$ for all x and t. Prove that if $\lim_{t \to t_0} f(x,t) = f(x,t_0)$ for every x, then $\lim_{t \to t_0} F(t) = F(t_0)$. In particular, if f is continuous in t for each fixed x, then F is continuous.

   (b) Suppose that $\partial f(x,t)/\partial t$ exists and there exists an integrable function g such that $|\partial f(x,t)/\partial t| \leq g(x)$ for all x and t. Prove that F is differentiable and

$$F'(t) = \frac{d}{dt} \int f(x,t)\, dx = \int \frac{\partial f(x,t)}{\partial t}\, dx.$$

   Hint: Use the dominated convergence theorem with any sequence $\{t_k\}$ in $[a,b]$ converging to t_0.