Math 8100 Assignment 2
Lebesgue measure and outer measure II

Due date: Friday the 5th of September 2014

1. (a) Let X be an uncountable set and $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$.
   i. Verify that $\mathcal{A}$ is a σ-algebra, called the *σ-algebra of countable or co-countable sets.*
   ii. Verify that the function μ on $\mathcal{M}$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E^c is countable is a measure on $\mathcal{A}$.

   (b) Let X be an infinite set. Defined $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Verify that μ is a finitely additive measure on $\mathcal{P}(X)$, but not a measure.

   (c) Let $(X, \mathcal{M})$ be a measurable space endowed with a measure μ and $E \in \mathcal{M}$. Define $\mu_E(F) = \mu(E \cap F)$ for all $F \in \mathcal{M}$. Prove that μ_E defines a measure on $\mathcal{M}$.

2. Let $E \subseteq \mathbb{R}$ with $m_*(E) = 0$. Show that the set $E^2 = \{x^2 \mid x \in E\}$ also Lebesgue outer measure zero.

   [*To what extent can you generalize this result?*]

3. (a) **The Borel-Cantelli Lemma.** Suppose $\{E_j\}_{j=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^n$ and that
   $$\sum_{j=1}^{\infty} m(E_j) < \infty.$$
   Let
   $$E = \limsup_{j\to\infty} E_j := \{x \in \mathbb{R}^n : x \in E_j, \text{ for infinitely many } j\}.$$
   Show that E is measurable and that $m(E) = 0$. *Hint: Write* $E = \cap_{k=1}^{\infty} \cup_{j\geq k} E_j$.

   (b) Given any irrational x one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions a/q, with a and q relatively prime integers, such that
   $$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$
   However, show that the set of those $x \in \mathbb{R}$ such that there exists infinitely many fractions a/q, with a and q relatively prime integers, such that
   $$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^3}$$
   is a set of Lebesgue measure zero.

4. Prove that any $E \subset \mathbb{R}$ with $m_*(E) > 0$ necessarily contains a non-measurable set.

5. This question deals with the G_δ and F_σ sets that were discussed in lecture (the definitions can also be found in the text).

   (a) Show that every closed set is a G_δ set and every open set is an F_σ set.
   Hint: If F is closed, consider $O_n = \{x : \inf_{y\in F} |x - y| < 1/n\}$.

   (b) Give an example of an F_σ set which is not a G_δ set.
   Hint: This is not so easy... You might want to use the Baire Category Theorem.

   (c) Give an example of a Borel set which is neither an F_σ nor a G_δ set.

6. A set $E \subseteq \mathbb{R}^n$ is said to be *Carathéodory measurable* if $m_*(A) = m_*(A \cap E) + m_*(E^c \cap A)$ for every set $A \subseteq \mathbb{R}^n$, where m_* denotes Lebesgue outer measure (as defined in class).

   Prove that a set $E \subseteq \mathbb{R}^n$ is *Lebegue measurable* if and only if E is *Carathéodory measurable.*

   Hint: If E is Lebesgue measurable and A is any set, choose a G_δ set V such that $A \subseteq V$ and $m_*(A) = m(V)$. Conversely, if E is Carathéodory measurable and $m_*(E) < \infty$, choose a G_δ set V with $E \subseteq V$ and $m_*(E) = m(V)$. Then $m_*(V \setminus E) = 0$.

Extra Challenge Problems

Not to be handed in with the assignment

1. If I is a bounded interval and $\alpha \in (0,1)$, let us call the open interval with the same midpoint as I and length equal to α times the length of I the "open middle αth" of I. If $\{\alpha_j\}_{j=1}^\infty$ is any sequence of numbers in $(0,1)$, then, we can define a decreasing sequence $\{K_j\}$ of closed sets as follows: $K_0 = [0,1]$, and K_j is obtained by removing the the open middle α_jth from each of the intervals that make up K_{j-1}. The resulting limiting set $K = \bigcap_{j=1}^\infty K_j$ is called a **generalized Cantor set**.

   (a) Suppose $\{\alpha_j\}_{j=1}^\infty$ is any sequence of numbers in $(0,1)$.

      i. Prove that $\prod_{j=1}^\infty (1 - \alpha_j) > 0$ if and only if $\sum_{j=1}^\infty \alpha_j < \infty$.

      ii. Given $\beta \in (0,1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_{j=1}^\infty (1 - \alpha_j) = \beta$.

   (b) Given $\beta \in (0,1)$, construct an open set G in $[0,1]$ whose boundary has Lebesgue measure β.

   Hint: Every closed nowhere dense set is the boundary of an open set.