Math 8100 Assignment 11

Abstract Measure and Integration

Due date: Wednesday the 10th of December 2014

Throughout this assignment $(X, \mathcal{M}, \mu)$ denotes a measure space with μ an *actual* measure.

1. Let X be a metric space and μ be a Borel measure on X that assigns a finite measure to all bounded Borel sets. Prove that if $\{F_n\}_{n=1}^{\infty}$ is a sequence of closed sets in X, then for any $\varepsilon > 0$ there exists a closed set $F \subseteq \bigcup_{n=1}^{\infty} F_n$ with the property that $\mu\big(\bigcup_{n=1}^{\infty} F_n \setminus F\big) < \varepsilon$.

2. Let $X = \mathbb{R}$ and $\mathcal{M} = \mathcal{B}_{\mathbb{R}}$, in other words let μ be a Borel measure on $\mathbb{R}$. Prove that if μ has the additional property that it is finite on all bounded Borel sets, then the function $F : \mathbb{R} \to \mathbb{R}$ defined for each $x \in \mathbb{R}$ by
$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu((x, 0]) & \text{if } x < 0, \end{cases}$$
is both increasing and right continuous on $\mathbb{R}$.

3. Suppose that $\mu(X) < \infty$ and $f \in L^1(X, \mathcal{M}, \mu)$.
   (a) Prove that if
   $$0 \leq \frac{1}{\mu(E)} \int_E f \, d\mu \leq 1$$
   for all $E \in \mathcal{M}$ with $\mu(E) > 0$, then $0 \leq f \leq 1$ almost everywhere.
   (b) Prove that if
   $$\left| \frac{1}{\mu(E)} \int_E f \, d\mu \right| \leq 1$$
   for all $E \in \mathcal{M}$ with $\mu(E) > 0$, then $|f| \leq 1$ almost everywhere.

4. Let ν be complex measure on $(X, \mathcal{M})$.
   (a) Prove that if ν is absolutely continuous with respect to μ if and only if its *total variation* $|\nu|$ is absolutely continuous with respect to μ, in other words verify that
   $$\nu \ll \mu \iff |\nu| \ll \mu.$$
   (b) Prove that $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ for all $E \in \mathcal{M}$ with $\mu(E) < \delta$.

5. Let ν be complex measure on $(X, \mathcal{M})$.
   (a) **(Polar representation of ν)** Prove that there exists a measurable function f such that $|f| = 1$ on X and
   $$\nu(E) = \int_E f \, d|\nu|$$
   for all $E \in \mathcal{M}$ (in other words $d\nu = f \, d|\nu|$).
   (b) Deduce from part (a) above that if $g \in L^1(X, \mathcal{M}, \mu)$ and
   $$\nu(E) = \int_E g \, d\mu$$
   for all $E \in \mathcal{M}$ (in other words $d\nu = g \, d\mu$), then
   $$|\nu|(E) = \int_E |g| \, d\mu,$$
   for all $E \in \mathcal{M}$ (in other words $d|\nu| = |g| \, d\mu$).

6. For review, prove the following standard results.

(a) i. **(Constructing outer measures)** Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of "elementary" sets and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. If we now define

$$\mu_*(E) = \inf\Big\{\sum_{j=1}^{\infty} \rho(E_j) \;:\; E_j \in \mathcal{E} \text{ and } E \subseteq \bigcup_{j=1}^{\infty} E_j\Big\}$$

for each $E \in \mathcal{P}(X)$, then μ_* defines an outer measure.

ii. **(Continuity from below)** If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n).$$

iii. **(Approximation by simple functions)** If $f \in L^+(X, \mathcal{M})$, then there exists a sequence $\{\phi_n\}_{n=1}^{\infty}$ of simple functions such that $0 \le \phi_1 \le \phi_2 \le \cdots \le f$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

iv. If $f \in L^+(X, \mathcal{M})$, then

$$\int f\, d\mu = 0 \iff f = 0 \text{ almost everywhere (with respect to the measure } \mu).$$

(b) i. **(Monotone Convergence Theorem)** If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^+(X, \mathcal{M})$ with $f_n(x)$ increasing to $f(x)$ for almost every x, then

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu.$$

ii. **(Interchanging sums and integrals of L^+ functions)** If $\{f_n\}_{n=1}^{\infty}$ is a finite or infinite sequence in $L^+(X, \mathcal{M})$, then

$$\int \Big(\sum_{n=1}^{\infty} f_n\Big)\, d\mu = \sum_{n=1}^{\infty} \Big(\int f_n\, d\mu\Big).$$

iii. **(Borel-Cantelli Lemma)** If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ with the property that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then almost all $x \in X$ lie in at most finitely many of the sets E_n. In other words

$$\mu(\limsup_{n\to\infty} E_n) = \mu\Big(\bigcap_{m=1}^{\infty} \bigcup_{n\ge m} E_n\Big) = 0.$$

iv. **(Fatou's Lemma)** If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^+(X, \mathcal{M})$, then

$$\int \liminf_{n\to\infty} f_n\, d\mu \le \liminf_{n\to\infty} \int f_n\, d\mu.$$

(c) i. If $f \in L^1(X, \mathcal{M}, \mu)$, then

$$\Big|\int f\, d\mu\Big| \le \int |f|\, d\mu.$$

ii. **(Dominated Convergence Theorem)** If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^1(X, \mathcal{M}, \mu)$ with the property that $f_n \to f$ almost everywhere and there exists a non-negative $g \in L^1(X, \mathcal{M}, \mu)$ such that $|f_n| \le g$ almost everywhere, then

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu.$$

iii. **(Interchanging sums and integrals of L^1 functions)** If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^1(X, \mathcal{M}, \mu)$ such that $\sum_{n=1}^{\infty} \Big(\int |f_n|\, d\mu\Big) < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges almost everywhere to a function in $L^1(X, \mathcal{M}, \mu)$, and

$$\int \Big(\sum_{n=1}^{\infty} f_n\Big)\, d\mu = \sum_{n=1}^{\infty} \Big(\int f_n\, d\mu\Big).$$

iv. Simple functions are dense in $L^1(X, \mathcal{M}, \mu)$ in the L^1-metric.

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that the total variation $|\nu|$ of a complex measure ν on $(X, \mathcal{M})$ is an actual measure on $(X, \mathcal{M})$ with the property that $|\nu|(X) < \infty$.

2. Let ν be a *real* measure on $(X, \mathcal{M})$.

   (a) **(Hahn Decomposition Theorem)** Prove that there exists sets $A, B \in \mathcal{M}$ such that $A \cup B = X$, $A \cap B = \emptyset$, and such that the positive and negative variations

   $$\nu^+ := \frac{|\nu| + \nu}{2} \qquad \text{and} \qquad \nu^- := \frac{|\nu| - \nu}{2}$$

   of ν satisfy

   $$\nu^+(E) = \mu(A \cap E) \qquad \text{and} \qquad \nu^-(E) = \mu(B \cap E)$$

   for all $E \in \mathcal{M}$.

   (b) **(Jordon Decomposition Theorem)** Deduce from part (a) above that if

   $$\nu = \mu_1 - \mu_2$$

   where μ_1 and μ_2 are actual measures on $(X, \mathcal{M})$, then $\mu_1 \geq \nu^+$ and $\mu_2 \geq \nu^-$.