Math 8100 Assignment 10
A Bit of Everything

Due date: Friday the 21st of November 2014

1. Let $X \subseteq \mathbb{R}^n$ be Lebesgue measurable. Prove that if $L^2(X) \subseteq L^1(X)$, then $m(X) < \infty$.

2. Let f and g be two non-negative Lebesgue measurable functions on $[0, \infty)$. Suppose that

$$A := \int_0^\infty f(y)\, y^{-1/2} dy < \infty \qquad \text{and} \qquad B := \left(\int_0^\infty |g(y)|^2 dy \right)^{1/2} < \infty$$

   Prove that

$$\int_0^\infty \left(\int_0^x f(y)\, dy \right) \frac{g(x)}{x}\, dx \leq AB$$

3. Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq 1$ for all $k \in \mathbb{N}$.
   (a) i. Prove that if $f_k \to f$ either a.e. on $[0,1]$ or in $L^1([0,1])$, then $f \in L^2([0,1])$ with $\|f\|_2 \leq 1$.
   ii. Do either of the above hypotheses guarantee that $f_k \to f$ in $L^2([0,1])$?
   (b) Prove that if $f_k \to f$ a.e. on $[0,1]$, then this in fact implies that $f_k \to f$ in $L^1([0,1])$.

4. For each $1 \leq p \leq \infty$, define $\Lambda_p : L^p([0,1]) \to \mathbb{R}$ by

$$\Lambda_p(f) = \int_0^1 x^2 f(x)\, dx.$$

   Explain why Λ_p is a continuous linear functional and compute its norm (in terms of p).

5. Let $1 \leq p \leq \infty$. Prove that if $\{f_k\}_{k=1}^\infty$ is a sequence of functions in $L^p(\mathbb{R}^n)$ with the property that

$$\sum_{k=1}^\infty \|f_k\|_p < \infty,$$

   then $\sum f_k$ converges almost everywhere to an $L^p(\mathbb{R}^n)$ function with

$$\left\| \sum_{k=1}^\infty f_k \right\|_p \leq \sum_{k=1}^\infty \|f_k\|_p.$$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Let g be a non-negative measurable function on X and $\nu_g : \mathcal{M} \to [0, \infty]$ be a set function defined for each $E \in \mathcal{M}$ by

$$\nu_g(E) := \int_E g\, d\mu$$

   Prove that ν_g defines a measure on $(X, \mathcal{M})$ which is absolutely continuous with respect to μ, and that

$$\int_X f\, d\nu_g = \int_X fg\, d\mu$$

   for any non-negative measurable function f on X.
   (b) Let $g \in L^1(X, \mu)$ and $\nu_g : \mathcal{M} \to \mathbb{C}$ be a set function defined for each $E \in \mathcal{M}$ by

$$\nu_g(E) := \int_E g\, d\mu.$$

   Prove that ν_g defines a complex measure on $(X, \mathcal{M})$ which is absolutely continuous with respect to μ.

Extra Challenge Problems

Not to be handed in with the assignment

If $f \in L^1(\mathbb{T}) := \{f \in L^1([0,1]) \,:\, f(0) = f(1)\}$, then the *Nth partial sum of the Fourier series of f*, is defined be

$$S_N f(x) = \sum_{|n|\leq N} \widehat{f}(n) e^{2\pi i n x}$$

where

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx,$$

for each $n \in \mathbb{Z}$. It is straightforward to see that one can re-express the Nth partial sums as follows:

$$S_N f(x) = f * D_N(x) := \int_0^1 f(y) D_N(x-y)\, dy$$

where

$$D_N(x) := \sum_{|n|\leq N} e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin \pi x} \qquad \text{(Dirichlet kernel).}$$

1. Consider the *Cesàro means* of the $S_N f$, namely

$$\sigma_N f := \frac{1}{N} \sum_{n=0}^{N-1} S_n f = f * F_N$$

where

$$F_N(x) := \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\Big(\frac{\sin(N\pi x)}{\sin \pi x}\Big)^2 \qquad \text{(Fejér kernel).}$$

(a) Verify that the Fejér kernel satisfies the following basic properties:

i. $0 \leq F_N(x) \leq C\dfrac{1}{N}\min\Big\{N^2, \dfrac{1}{|x|^2}\Big\}$ for some constant $C > 0$ and all $x \in [0,1]$,

ii. $\displaystyle\int_0^1 F_N(x)\, dx = 1$,

iii. $\displaystyle\lim_{N\to\infty} \int_{\delta \leq |x| \leq \frac{1}{2}} F_N(x)\, dx = 0$ for any choice of $\delta > 0$.

[Note also that $\widehat{F_N}(n) = \max\Big\{1 - \dfrac{|n|}{N}, 0\Big\}$ for all $n \in \mathbb{Z}$.]

(b) Use the *approximation to the identity*-type properties above to prove the following

Theorem 1 (Fejér's Theorem)**.** *Let* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

(i) (Classical version) *If* $f \in C(\mathbb{T})$, *then* $\sigma_N f \to f$ *uniformly on* $\mathbb{T}$ *as* $N \to \infty$.

(ii) (L^1-version) *If* $f \in L^1(\mathbb{T})$, *then* $\sigma_N f \to f$ *in* $L^1(\mathbb{T})$ *as* $N \to \infty$.

[It is also true that if $f \in L^p(\mathbb{T})$ with $1 \leq p < \infty$, then $\sigma_N f \to f$ in $L^p(\mathbb{T})$ as $N \to \infty$.]

(c) Verify that Theorem 1 gives a new proof that *Trigonometric polynomials are dense in both* $C(\mathbb{T})$ *and in* $L^1(\mathbb{T})$, and that Theorem 1 (ii) in particular has the following important (new) consequence:

Corollary 1.

If $f \in L^1(\mathbb{T})$ *and* $\widehat{f}(n) = 0$ *for all* $n \in \mathbb{Z}$, *then* $f(x) = 0$ *for almost every* $x \in \mathbb{T}$.

2. Use Corollary 1 above to prove the following strengthening of Question 6 on Homework Assignment 9:

Theorem 2 (Periodic analogue of the Fourier inversion formula)**.**

If $f \in L^1(\mathbb{T})$ *and* $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$, *then* $S_N f(x) \to f(x)$ *for almost every* $x \in \mathbb{T}$ *as* $N \to \infty$.