Math 8100 Assignment 1
Lebesgue measure and outer measure

Due date: Wednesday the 27th of August 2014

1. The **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Thus $\mathcal{C}$ is obtained from $[0,1]$ by removing the open middle third $(\frac{1}{3}, \frac{2}{3})$, then removing the open middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the two remaining intervals, and so forth.

   (a) Find a real number x belonging to the Cantor set which is not the endpoint of one of the intervals used in its construction.

   (b) Prove that $\mathcal{C}$ is compact, nowhere dense, totally disconnected, and perfect.

   (c) Prove that $m_*(\mathcal{C}) = 0$.

   (d) Prove that $\mathcal{C}$ is uncountable by showing that the function $f(x) = \sum_{k=1}^{\infty} b_k 2^{-k}$ where $b_k = a_k/2$, maps $\mathcal{C}$ onto $[0,1]$.

2. Prove that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

3. Suppose that $A \subseteq E \subseteq B$, where A and B are Lebesgue measurable subsets on $\mathbb{R}^n$.

   (a) Prove that if $m(A) = m(B) < \infty$, then E is measurable.

   (b) Give an example showing that the same conclusion does not hold if A and B have infinite measure.

4. Suppose A and B are a pair of compact subsets of $\mathbb{R}^n$ with $A \subseteq B$, and let $a = m(A)$ and $b = m(B)$. Prove that for any c with $a < c < b$, there is a compact set E with $A \subseteq E \subseteq B$ and $m(E) = c$.

   Hint: As an example, if $n = 1$ and E is a measurable subset of $[0,1]$, consider $m(E \cap [0,t])$ as a function of t.

5. The **outer Jordan content** $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|,$$

   where the infimum is taken over every *finite* covering $E \subseteq \cup_{j=1}^{N} I_j$, by intervals I_j.

   (a) Prove that $J_*(E) = J_*(\bar{E})$ for every set E (here $\bar{E}$ denotes the closure of E).

   (b) Exhibit a countable subset $E \subseteq [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

6. Let $\mathcal{N}$ denote the non-measurable subset of $[0,1]$ that was constructed in lecture.

   (a) Prove that if E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

   (b) Show that $m_*([0,1] \setminus \mathcal{N}) = 1$
   [*Hint: Argue by contradiction and pick an open set G such that $[0,1] \setminus \mathcal{N} \subseteq G \subseteq [0,1]$ with $m_*(G) \leq 1 - \varepsilon$.*]

   (c) Conclude that there exists *disjoint* sets $E_1 \subseteq [0,1]$ and $E_2 \subseteq [0,1]$ for which

$$m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2).$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Let $\mathcal{C}$ denote the usual (middle-third) Cantor set. Prove that $\mathcal{C} + \mathcal{C} = [0, 2]$.

   [*Hint: Consider the intersection of the set* $\mathcal{C} \times \mathcal{C} \subset \mathbb{R}^2$ *and the family of lines* $\{x + y = c \,|\, c \in [0, 2]\}$ *and use the property of nested compact sets.*]

2. Complete the following outline to prove that a bounded function on an interval $[a, b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.

   Let f be a bounded function on a compact interval $[a, b]$ and

   $$\operatorname{osc}(f, c) = \lim_{\delta \to 0} \sup_{x,y \in B_\delta(c) \cap [a,b]} |f(x) - f(y)|$$

   define the oscillation of f at c. Clearly, f is continuous at $c \in [a, b]$ if and only if $\operatorname{osc}(f, c) = 0$.

   (a) Let $A_\varepsilon = \{c \in [a, b] \,:\, \operatorname{osc}(f, c) \geq \varepsilon\}$. Prove that for every $\varepsilon > 0$, the set A_ε is compact.

   (b) Prove that if the set of discontinuities of f has measure zero, then f is Riemann integrable.

   [*Hint: Let* $\varepsilon > 0$. *Cover* A_ε *by a finite number of open intervals whose total length is* $\leq \varepsilon$. *Select and appropriate partition of* $[a, b]$ *and estimate the difference between the upper and lower sums of* f *over this partition.*]

   (c) Prove that if f is Riemann integrable on $[a, b]$, then its set of discontinuities has measure zero.

   [*Hint: The set of discontinuities of* f *is contained in* $\bigcup_n A_{1/n}$. *Given* $\varepsilon > 0$, *choose a partition* P *such that* $U(f, P) - L(f, P) < \varepsilon/n$. *Show that the total length of the intervals in* P *whose interiors intersect* $A_{1/n}$ *is* $\leq \varepsilon$.]