Proof of Fubini & Tonelli's Theorems

Proof of Fubini

In order to prove Fubini we will consider a series of special cases:

Let
$$\mathcal{F} = \{ f \in L^1(\mathbb{R}^n) : \text{Fubini's thm holds for } f \}.$$

Case Jumping Lemma

(i) Finite linear combinations of functions in $\mathcal{F}$ remain in $\mathcal{F}$.

(ii) If $\{f_k\} \subseteq \mathcal{F}$ & $f_k \nearrow f$ or $f_k \searrow f$ (with $f \in L^1$) $\Rightarrow f \in \mathcal{F}$.

Proof:

(i): Follows by linearity.

(ii): By replacing f_k with $-f_k$ we may assume $\{f_k\}$ is increasing.
By replacing f_k with $f_k - f_1$ we may assume that $f_k \nearrow f$ & $f_k \geq 0$.

For each k, $\exists\, N_k \in \mathcal{M}(\mathbb{R}^{n_1})$ s.t. $m(N_k) = 0$ and $(f_k)_x \in L^1(\mathbb{R}^{n_2})\ \forall x \notin N_k$.

Let $N = \bigcup N_k$. Notice that $m(N) = 0$ and $(f_k)_x \in L^1(\mathbb{R}^{n_2})\ \forall k, \forall x \notin N$

$$\text{MCT} \Rightarrow \int_{\mathbb{R}^{n_2}} f_k(x,y)\,dy \nearrow \int_{\mathbb{R}^{n_2}} \underbrace{f(x,y)}_{\text{integrable}}\,dy \quad \forall x \notin N$$

$$\text{MCT} \Rightarrow \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f_k(x,y)\,dy \right) d\cancel{y}x \longrightarrow \int_{\mathbb{R}^{n_1}} \left(\underbrace{\int_{\mathbb{R}^{n_2}} f(x,y)\,dy}_{\text{int'ble } \& < \infty \text{ a.e.}} \right) dx$$

$f_k \in \mathcal{F}$ $\longrightarrow$ $\Vert$ $\qquad$ $\mathbin{/\!/}$ $\longleftarrow$ By uniqueness of limits these are equal!

$$\text{MCT} \Rightarrow \int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f$$

(*) Work back to see $\int_{\mathbb{R}^{n_2}} f\,dy$ & f_x int'ble $\square$.

In light of the "Case Jumping Lemma" we see that the proof of Fubini's theorem reduces to:

Claim

If $E \in \mathcal{M}(\mathbb{R}^n)$ with $m(E) < \infty \not\Rightarrow \chi_E \in \mathcal{F}$.

Since any $E \in \mathcal{M}(\mathbb{R}^n)$ can be expressed as

$$E = V \setminus N \qquad (\Rightarrow \chi_E = \chi_V - \chi_N)$$

with V a G_δ-set and $m(N) = 0$, the Claim will follow from:

Lemma 1

If V is a G_δ-set with $m(V) < \infty$, then $\chi_V \in \mathcal{F}$.

Lemma 2

If $N \subseteq \mathbb{R}^n$ with $m(N) = 0$, then $\chi_N \in \mathcal{F}$.

Proof of Lemma 2 (assuming Lemma 1)

Since $N \subseteq V_1$ with V_1 a G_δ-set with $m(V_1) = 0$

Lemma 1 $\Rightarrow$ $0 = \int \chi_{V_1} = \int \left(\int \chi_{V_1} \, dy \right) dx$

$\Rightarrow \int \chi_{V_1} \, dy = 0$ a.e. x

$\Rightarrow \int \chi_N \, dy = 0$ a.e x (since $N \subseteq V_1$)

$\Rightarrow \int \left(\int \chi_N \, dy \right) dx = 0$

But $m(N) = \int_{\mathbb{R}^n} \chi_N = 0$ also, so $\chi_N \in \mathcal{F}$. $\square$

Proof of Lemma 1 : V is a G_δ-set so $V = \bigcap_{j=1}^{\infty} G_j$ with G_j open.

Without loss in generality we will assume that $G_1 \supseteq G_2 \supseteq \cdots$ & $m(G_1) < \infty$.

[o/w replace $\{G_j\}$ with $\{G_1, G_1 \cap G_2, G_1 \cap G_2 \cap G_3, \ldots\}$]

Then $\chi_{G_j} \searrow \chi_V$ and by the "Case Jumping Lemma" matters reduce to

SubClaim 1 : G open with $m(G) < \infty \Rightarrow \chi_G \in \mathcal{F}$.

But any open set $G = \bigcup_{j=1}^{\infty} Q_j$ with $\{Q_j\}$ disjoint, partially open cubes.

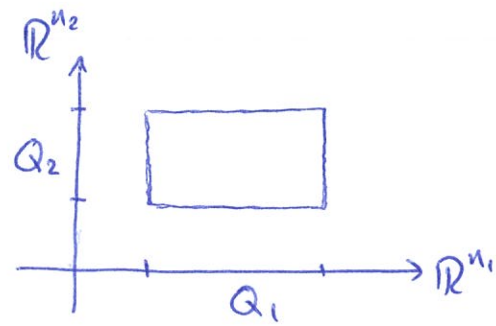
~~and hence~~ $\cancel{\chi_{Q_j} \nearrow \chi_G}$

Hence if we define $G_K = \bigcup_{j=1}^{K} Q_j$, then $\chi_{G_K} \nearrow \chi_G$ & $\chi_{G_K} = \chi_{Q_1} + \cdots + \chi_{Q_K}$

so by the "Case Jumping Lemma" matters reduce to

SubClaim 2 : Q bounded partially open cube $\Rightarrow \chi_Q \in \mathcal{F}$.

Proof of SubClaim 2 :



- Suppose $Q \subseteq \mathbb{R}^n$ is an open cube.
  Note that $Q = Q_1 \times Q_2 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$
  and $\int_{\mathbb{R}^n} \chi_Q = m(Q) = m(Q_1) m(Q_2)$.

For a.e. $x \in \mathbb{R}^{n_1}$ $\chi_Q(x,y)$ is int'ble in y & $\int \chi_Q \, dy = \begin{cases} m(Q_2) & \text{if } x \in Q_1 \\ 0 & \text{o/w} \end{cases} = \chi_{Q_1} m(Q_2)$

$$\Rightarrow \int \left(\int \chi_Q \, dy \right) dx = \int \chi_{Q_1}(x) m(Q_2) \, dx = m(Q_1) m(Q_2) \quad \checkmark$$

- Suppose $E \subseteq$ bdry of closed cube in $\mathbb{R}^n$. Then for a.e x
  $E_x = \{y : (x,y) \in E\}$ has measure zero!

$$\Rightarrow \int_{\mathbb{R}^{n_2}} \chi_E(x,y) \, dy = 0 \quad \text{a.e. } x \quad \Rightarrow \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} \chi_E(x,y) \, dy \right) dx = 0$$

Since $\int_{\mathbb{R}^n} \chi_E = m(E) = 0$ $\checkmark$

$\square$

Proof of Tonelli's Theorem (Assuming Fubini)

Let $f_k(x,y) = \begin{cases} f(x,y) & \text{if } |(x,y)| \leq k \ \& \ f(x,y) \leq k. \\ 0 & \text{o/w}. \end{cases}$

Each $f_k \geq 0$ and integrable (so Fubini applies) & $f_k \nearrow f$. (*)

- For a.e. $x \in \mathbb{R}^{n_1}$:

$$\underbrace{(f_k)_x(y) = f_k(x,y)}_{\substack{\text{all m'ble fns of } y \\ \text{(since they are int'ble)}}} \nearrow f_x(y) = f(x,y)$$

is measurable as a function of y on $\mathbb{R}^{n_2}$.

- For a.e. $x \in \mathbb{R}^{n_1}$: It follows from the MCT that

$$\int_{\mathbb{R}^{n_2}} f_k(x,y)\,dy \nearrow \int_{\mathbb{R}^{n_2}} f(x,y)\,dy \qquad (**)$$

and since this is a m'ble function of x it follow that $\int_{\mathbb{R}^{n_2}} f(x,y)\,dy$ is also.

- Applying the MCT ~~one~~ two more time gives:

(i) $\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f_k(x,y)\,dy \right) dx \longrightarrow \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx$

(because of (**))

& (ii) $\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f$ (because of (*))

Result follows by uniqueness of limits, since $\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f_k\,dy \right) dx = \int_{\mathbb{R}^n} f_k \ \forall k$ by Fubini

$\square$