Repeated Integration : Fubini & Tonelli's Theorems

Fubini's Theorem

"Finiteness of multiple int $\Rightarrow$ finiteness of all iterated ints (& all equal)".

Let $f(x,y)$ be Lebesgue int'ble on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for a.e. $x \in \mathbb{R}^{n_1}$

(i) $f_x(y) = f(x,y)$ is an int'ble function of y on $\mathbb{R}^{n_2}$

(ii) $\int_{\mathbb{R}^{n_2}} f(x,y)\,dy$ is an int'ble function of x on $\mathbb{R}^{n_1}$

Moreover,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^n} f \quad .$$

In order to fully benefit from Fubini's theorem (using it "positively") we need a viable way to check that function are integrable.

Tonelli's Theorem

"For $f \geq 0$: Finiteness of any one of Fubini's 3 ints $\Rightarrow$ Finiteness of other two!"

Let $f(x,y)$ be non-negative and measurable on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for a.e. $x \in \mathbb{R}^{n_1}$

(i) $f_x(y) = f(x,y)$ is measurable as a function of y on $\mathbb{R}^{n_2}$

(ii) $\int_{\mathbb{R}^{n_2}} f(x,y)\,dy$ is measurable as a function of x on $\mathbb{R}^{n_1}$

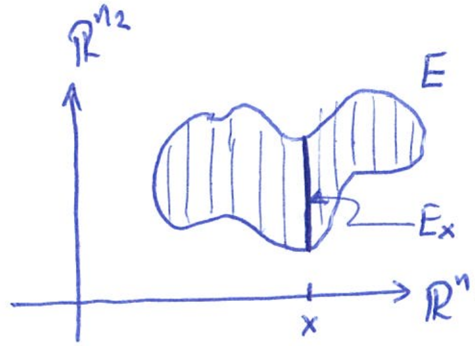
Moreover,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^n} f \quad .$$

Corollary (of Tonelli)

If E is a Lebesgue measurable subset of $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for a.e. $x \in \mathbb{R}^{n_1}$ the "slice" $E_x := \{y \in \mathbb{R}^{n_2} : (x,y) \in E\}$ is a Lebesgue measurable subset of $\mathbb{R}^{n_2}$ and $m(E_x)$ is a measurable function of x in $\mathbb{R}^{n_1}$. Moreover,

$$\int_{\mathbb{R}^{n_1}} m(E_x)\,dx = m(E).$$



¿ Is it true that if for a given set $E \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ we knew that for a.e. $x \in \mathbb{R}^{n_1}$ that the slices E_x were m'ble subsets of $\mathbb{R}^{n_2}$, then E measurable in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$?

NO! Consider $E = [0,1] \times \mathcal{N} \Rightarrow E^y := \{x \in \mathbb{R}^{n_1} : (x,y) \in E\}$

$$= \begin{cases} [0,1] & \text{if } y \in \mathcal{N} \\ \phi & \text{o/w} \end{cases} \in \mathcal{M}(\mathbb{R}^{n_1}).$$

So <u>if</u> $E \in \mathcal{M}(\mathbb{R}^n)$, Corollary $\Rightarrow E_x \in \mathcal{M}(\mathbb{R}^{n_2})$, but $E_x = \mathcal{N}$ ↯.

Remark: In practice we often combine Fubini & Tonelli as follows:

Let $f(x,y)$ be m'ble on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. If either

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} |f(x,y)|\,dy \right) dx \quad \text{or} \quad \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} |f(x,y)|\,dx \right) dy$$

is finite, then $f \in L^1(\mathbb{R}^n)$ (by Tonelli applied to $|f(x,y)|$), thus $\int_{\mathbb{R}^n} f < \infty$ and (by Fubini) we know that

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} f(x,y)\,dx \right) dy$$

Two Examples (using Fubini to show function are non-integrable)

Example 1

Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$ on $[0,1]\times[0,1]$.

Since

$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3}\, dy \right) dx = \frac{1}{2} \quad (\underrightarrow{\text{Exercise}})$$

we also have that

$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3}\, dx \right) dy = -\frac{1}{2}$$

and Fubini $\Rightarrow f \notin L^1([0,1]\times[0,1])$.

Example 2 (converse of Fubini false!)

Let $f(x,y) = \dfrac{xy}{(x^2+y^2)^2}$ on $[-1,1]\times[-1,1]$.

It is immediately clear that

$$\int_{-1}^{1} f(x,y)\, dx = \int_{-1}^{1} f(x,y)\, dy = 0$$

and hence that both iterated integrals equal 0.

However,

$$\int_{-1}^{1}\left(\int_{-1}^{1} |f(x,y)|\, dx\right) dy \overset{\text{Exercise}}{=} 2\int_0^1 \left(\frac{1}{y} - \frac{y}{1+y^2}\right) dy \text{ which } \underline{\text{DNE}}!$$

Fubini $\Rightarrow |f| \notin L^1([-1,1]\times[-1,1]) \Leftrightarrow f \notin L^1([-1,1]\times[-1,1])$.

Appendix (on Measurability on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$).

Lemma

If f measurable on $\mathbb{R}^{n_1}$, then $F(x,y)=f(x)$ is measurable on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Proof: Assume that $n_2=1$. Need to show that for all $a\in\mathbb{R}$

$$\{(x,y)\in\mathbb{R}^n\times\mathbb{R} : F(x,y)>a\}\in\mathcal{M}(\mathbb{R}^{n+1}).$$
$$\|$$
$$\{x\in\mathbb{R}^n : f(x)>a\}\times\mathbb{R}$$

* Things thus reduce to showing that if $E\in\mathcal{M}(\mathbb{R}^n)$, then $E\times\mathbb{R}\in\mathcal{M}(\mathbb{R}^{n+1})$:

- Write $E=H\cup N$ with H a F_σ-set and $m(N)=0$.

  $\Rightarrow\ E\times\mathbb{R}=(H\times\mathbb{R})\cup(N\times\mathbb{R})$.

  Since $H\times\mathbb{R}$ is clearly a F_σ-set in $\mathbb{R}^{n+1}$ we will be done if we can show that $N\times\mathbb{R}$ has measure zero in $\mathbb{R}^{n+1}$:

- Define $E_k=\{x\in\mathbb{R} : |x|\le k\}$, then $E_1\subseteq E_2\subseteq\cdots$ & $\bigcup_k E_k=\mathbb{R}$.

  $\Rightarrow\ N\times E_1\subseteq N\times E_2\subseteq\cdots$ and $\bigcup_k(N\times E_k)=N\times\mathbb{R}$.

  and hence that $m(N\times\mathbb{R})=\lim_{k\to\infty} m(N\times E_k)=0$ □

  Claim: For each $k\in\mathbb{N}$, $m(N\times E_k)=0$.

  Pf: Fix k & let $\varepsilon>0$. Since N is null in $\mathbb{R}^n$ we know that

  $N\subseteq\bigcup_j Q_j$ with $\sum_j|Q_j|<\varepsilon/2k$. (with $\{Q_j\}$ closed cubes)

  $\Rightarrow\ N\times E_k\subseteq\bigcup_j(Q_j\times E_k)$ with $\sum_j|Q_j\times E_k|=\sum_j 2k|Q_j|<\varepsilon$ □

  $\uparrow$ cubes!

Consequence of Lemma 1

① f & g m'ble on $\mathbb{R}^{n_1}$ & $\mathbb{R}^{n_2}$ $\Rightarrow$ $H(x,y) = f(x)g(y)$ m'ble on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$

[$H(x,y) = F(x,y)G(x,y)$ where $F(x,y) = f(x)$ & $G(x,y) = g(y)$.]

② f, g m'ble on $\mathbb{R}^n$ $\Rightarrow$ $h(x,y) = f(x-y)g(y)$ m'ble on $\mathbb{R}^{2n}$.

[$h(x,y) = F\circ T(x,y)\, G(x,y)$ where $F(x,y) = f(x)$, $G(x,y) = g(y)$
$= F(x-y, x+y)\, G(x,y)$ and $\underline{T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$]

③ $f \geq 0$ & m'ble on $\mathbb{R}^n$ $\Rightarrow$ $\tilde{F}(x,y) = y - f(x)$ m'ble on $\mathbb{R}^{n+1}$ for any $y \in \mathbb{R}$.

[$\tilde{F}(x,y) = G(x,y) - F(x,y)$ where $G(x,y) = y$ & $F(x,y) = f(x)$]

"Area under Graph"

Suppose $f(x) \geq 0$ on $\mathbb{R}^n$ & $\mathcal{A} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x)\}$, then

(i) f m'ble on $\mathbb{R}^n$ $\Leftrightarrow$ $\mathcal{A} \in \mathcal{M}(\mathbb{R}^{n+1})$

(ii) f m'ble on $\mathbb{R}^n$ $\Rightarrow$ $\int_{\mathbb{R}^n} f(x)\,dx = m(\mathcal{A})$.

__Proof__: __(i)__: ($\Rightarrow$) follows from ③ since $\mathcal{A} = \{y \geq 0\} \cap \{\tilde{F} \leq 0\}$

($\Leftarrow$) Corollary of Tonelli $\Rightarrow$ $f(x) = m(\mathcal{A}_x)$ is m'ble.

(ii) Corollary of Tonelli $\Rightarrow$ $m(\mathcal{A}) = \int_{\mathbb{R}^n} m(\mathcal{A}_x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx$. □