Cesàro Means & Fejér's Theorem

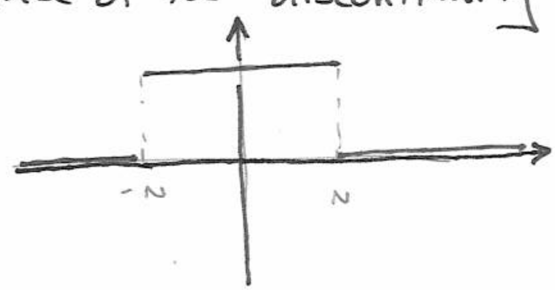
In order to recover a function f from its Fourier coefficients it would be convenient to find some other method than taking the limit of $S_N f$ since, as we have seen, this approach does not always work well.

Recall that $S_N f(x) \to f(x)$ fails to hold for all $x \in \mathbb{T}$ if f is merely continuous on $\mathbb{T}$. The difficulty with the operator $S_N f$, namely the fact that

$$\int_0^1 |D_N(t)|\,dt \to \infty \quad \text{as } N \to \infty$$

can be regarded as a consequence of the "discontinuity" of

$$\widehat{D_N}(n) = \chi_{\{-N,\dots,N\}}(n)$$



We may therefore hope to get an operator which is easier to analyze if we replace $D_N(x)$ with a suitable average whose Fourier coefficients do not exhibit such "jumps".

Arithmetic mean of the partial sums.

One elementary way to do this: Cesàro Means

As you all know $\lim_{n\to\infty} a_n$ exists $\Rightarrow \lim_{n\to\infty} \frac{a_1 + \dots + a_n}{n}$ also exists (& has same value).

Exercise 2: Show that the converse is false.

Define the Cesàro Mean of $S_N f$ to be

$$\sigma_N f := \frac{1}{N} \sum_{n=0}^{N-1} S_N f .$$

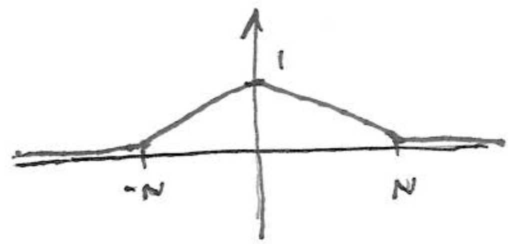
Setting $F_N := \frac{1}{N} \sum_{n=0}^{N-1} D_N$ ← Fejēr kernel.

it is easy to see that $\sigma_N f = f * F_N$.

Exercise 3

(a) $\hat{F}_N(n)$ looks like a triangle: $\hat{F}_N(n) = \left(1 - \frac{|n|}{N}\right)_+$ for all $n \in \mathbb{Z}$

(b) $F_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin \pi x} \right)^2$



(c) $0 \leq F_N(x) \leq \frac{1}{N} C \min\{N^2, \frac{1}{|x|^2}\}$

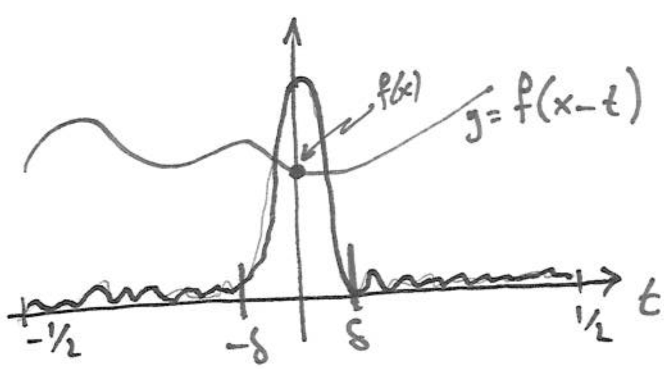
& $\int_0^1 F_N(x)dx = 1$.

(d) For every $\delta > 0$, $\int_{\delta \leq |t| \leq \frac{1}{2}} F_N(x)\,dx \to 0$ as $N \to \infty$

Theorem 2 (Fejér's Thm)

Let $1 \leq p \leq \infty$. If $f \in L^p(\mathbb{T})$, or $f \in C(\mathbb{T})$ if $p = \infty$, then

$$\lim_{N\to\infty} \|\sigma_N f - f\|_p = 0 .$$



(In particular, $\sigma_N f \to f$ uniformly whenever $f \in C(\mathbb{T})$.)

Proof of Fejér's Theorem

$$\| f * F_N - f \|_p \underset{\uparrow}{\leq} \int_{-1/2}^{1/2} |F_N(t)| \, \| \tau_t f - f \|_p \, dt \qquad (\tau_t f(x) = f(x-t))$$

Minkowski & fact that $\int_0^1 F_N(x)\,dx = 1$.

Recall that $\|\tau_t f - f\|_p \to 0$ as $t \to 0$ if $1 \leq p < \infty$ & $p = \infty$ <u>if</u> f also conts.

Hence

$$\| f * F_N - f \|_p \leq \underbrace{\int_{|t| \leq \delta} \text{—''—}}_{\downarrow \; 0 \text{ as } \delta \to 0.} + \underbrace{\int_{\delta \leq |t| \leq \frac{1}{2}} 2 \|f\|_p |F_N(t)| \, dt}_{\downarrow \; 0 \text{ as } N \to \infty \text{ for any fixed } \delta > 0.} .$$

□

<u>Corollary 1</u>:

- Trig polys are dense in $L^p(\mathbb{T})$, $1 \leq p < \infty$.
- Continuous functions on $\mathbb{T}$ can be uniformly approximated by trig polys.

<u>Corollary 2</u>:

If $f \in L^1(\mathbb{T})$ & $\hat{f}(n) = 0 \ \forall n \in \mathbb{Z}$, then $f = 0$ a.e.

* A slightly more difficult result (its proof uses the Hardy-Littlewood maximal function theorem) is the following.

<u>Theorem 3</u>: If $f \in L^1(\mathbb{T})$, then $\sigma_N f \to f$ a.e.