Fourier Series: Convergence and Summability

To any $f \in L^1(\mathbb{T})$ we associate its Fourier series

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

where, for each $n \in \mathbb{Z}$, the nth Fourier coefficient

$$\hat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x}\, dx .$$

* The central question that we will explore in this note is the following:

¿When, and in what sense, is f "equal" to its Fourier series?

Note: If f is a trigonometric polynomial: $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$

where all but finitely many of the a_n's are zero, then $a_n = \hat{f}(n)$.

In other words, $f(x) = \sum'_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$

[¿Why?: By orthogonality, namely $\int_0^1 e^{2\pi i n x}\, dx = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \in \mathbb{Z}\setminus\{0\} \end{cases}$.]

It is therefore natural (?) to explore the question of convergence of the Fourier series for more general functions. To answer such questions one must of course specify the sense of convergence.

We start with the most classical situation:

Pointwise Convergence

We introduce the Dirichlet summation operators

$$S_N f(x) = \sum_{|n| \le N} \hat{f}(n) e^{2\pi i n x}.$$

Note:

$$S_N f(x) = \sum_{|n| \le N} \int_0^1 f(y) e^{-2\pi i n y}\, dy \; e^{2\pi i n x}$$

$$= \int_0^1 f(y) \sum_{|n| \le N} e^{2\pi i n (x-y)}\, dy$$

$$=: \int_0^1 f(y) D_N(x-y)\, dy = \underline{f * D_N(x)}$$

where

$$D_N(x) = \sum_{|n| \le N} e^{2\pi i n x} \qquad \text{(Dirichlet Kernel)}$$

Exercise 1

(a) Verify that for each $N \geq 0$,

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

Oscillatory nature & growth rate is an indication that it may be in general, very delicate to understand the convergence properties of $S_N f$

(b) For all $N \geq 1$,

$$|D_N(x)| \le C \min\left\{N, \frac{1}{|x|}\right\}$$

(c) For all $N \geq 2$,

$$c \log N \le \int_0^1 |D_N(x)|\, dx \le C \log N, \text{ while } \int_0^1 D_N(x)\, dx = 1 \ \ \forall N \geq 0.$$

Theorem 1 (Dini) If, for some $x \in \mathbb{T}$, $\exists\, \delta > 0$ such that

$$(*) \quad \int_{|t| \leq \delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty$$

then $S_N f(x) \longrightarrow f(x)$.

Note: If f is Hölder continuous at x (i.e. if $|f(x+t) - f(x)| \leq C|t|^a$, for some $a > 0$) then f satisfies $(*)$ for some $\delta > 0$.

But, continuous functions need not satisfy $(*)$, in fact:

Theorem 2 (Du Bois-Reymond)

There exist continuous functions on $\mathbb{T}$ whose Fourier series diverges at a point!

Remark: It is perhaps striking that Dini's theorem is a "local result"; the convergence of $S_N f(x)$ depends only on the behaviour of f near x, but if we modify f away from x, this would change the Fourier coefficients and hence the series $S_N f(x)$.

Corollary (of Theorem 1) [Riemann Localization Theorem]

If $f \equiv 0$ in a neighbourhood of x, then $S_N f(x) \to 0$.

Before embarking on a proof of Dini's theorem we will discuss the so-called Riemann-Lebesgue lemma.

The Riemann-Lebesgue Lemma

Clearly, for any $f \in L^1(\mathbb{T})$, $|\hat{f}(n)| \leq \int_0^1 |f(x)|\,dx \;\; \forall\, n \in \mathbb{Z}$.

(i.e. $\wedge : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$)

In fact:

Lemma (Riemann-Lebesgue) $f \in L^1(\mathbb{T}) \Rightarrow \lim_{|n|\to\infty} \hat{f}(n) = 0$

Proof: Let $\varepsilon > 0$. Since $f \in L^1(\mathbb{T})$ we know $\exists\, g \in C(\mathbb{T})$ such that

$$\|f-g\|_{L^1} < \varepsilon/2 \quad \text{and hence} \quad \|\hat{f}-\hat{g}\|_{\ell^\infty} < \varepsilon/2 .$$

Since

$$\begin{aligned}\hat{g}(n) &= \int_0^1 g(x) e^{-2\pi i n x}\,dx \\ &= -\int_0^1 g(x) e^{-2\pi i n x} e^{-\pi i}\,dx \\ &= -\int_0^1 g(x) e^{-2\pi i n\left(x+\frac{1}{2n}\right)}\,dx \\ &= -\int_0^1 g\left(x-\tfrac{1}{2n}\right) e^{-2\pi i n x}\,dx\end{aligned}$$

since $e^{2\pi i n x}$ & $g(x)$ have period 1. (they are fns on $\mathbb{T}$).

it follows that

$$\hat{g}(n) = \frac{1}{2}\int_0^1 \left[g(x) - g\left(x-\tfrac{1}{2n}\right)\right] e^{-2\pi i n x}\,dx .$$

Since g is uniformly cont's, we know $\exists\, N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow \left|g(x) - g\left(x-\tfrac{1}{2n}\right)\right| < \varepsilon \text{ \& hence } |\hat{g}(n)| < \tfrac{\varepsilon}{2} .$$

Thus, if $n \geq N \Rightarrow |\hat{f}(n)| \leq |\hat{f}(n)-\hat{g}(n)| + |\hat{g}(n)| < \varepsilon$. $\square$

Corollary: If $g \in L^1(\mathbb{T})$, then $\lim_{N\to\infty} \int_0^1 g(t)\sin((2N+1)\pi t)\,dt = 0$.

Proof of Dini's Criterion (Theorem 1)

Recall that

$$S_N f(x) = \int_0^1 f(x-t)\, D_N(t)\, dt$$

where

$$D_N(t) = \frac{\sin((2N+1)\pi t)}{\sin \pi t} \quad \text{satisfies} \quad \int_0^1 D_N(t)\, dt = 1 .$$

Hence

$$S_N f(x) - f(x) = \int_{-1/2}^{1/2} \left[f(x-t) - f(x) \right] \frac{\sin((2N+1)\pi t)}{\sin \pi t}\, dt$$

δ from assumption

$$= \int_{|t| \le \delta} g(t) \sin((2N+1)\pi t)\, dt + \int_{\delta < |t| \le \frac{1}{2}} g(t) \sin((2N+1)\pi t)\, dt$$

where $\quad g(t) = \dfrac{f(x-t) - f(x)}{\sin \pi t}$.

By the corollary to Riemann-Lebesgue, it suffices to establish:

$$\text{(i) } \chi_{\{|t| \le \delta\}}\, g \in L^1(\mathbb{T}) \quad \& \quad \text{(ii) } \chi_{\{\delta < |t| \le \frac{1}{2}\}}\, g \in L^1(\mathbb{T}).$$

(i): From assumption, since if $|t| \le \delta$, then $\sin \pi t \approx \pi t$.

(ii): $\left| \chi_{\{\delta < |t| \le \frac{1}{2}\}}(t)\, g(t) \right| \le \dfrac{|f(x-t)| + |f(x)|}{\sin \delta\pi}$ ← this is in $L^1(\mathbb{T})$.

$\square$

Proof of Du Bois-Reymond's Theorem

This follows almost immediately from the Uniform Boundedness Principle and the fact that $\int_0^1 |D_N(t)|\,dt \to \infty$ as $N \to \infty$ (Exercise 1).

Recall: Uniform Boundedness Principle

Let X be a Banach space & Y be a normed vector space. If $T_\alpha : X \to Y$ is a family of bounded linear operators with the property that $\sup_\alpha \|T_\alpha x\|_Y < \infty$ for all $x \in X$, then

$$\sup_\alpha \|T_\alpha\|_{\mathcal{L}(X,Y)} < \infty.$$

* This is magic! We get uniform bounds from pointwise ones!! *

[For a proof, see Folland p.163 (it follows from the Baire Category Theorem)]

Proof of Theorem 2

Exercise 2

Consider the family of operators

$$f \longmapsto S_N f(0) = \int_0^1 f(t) D_N(t)\,dt$$

Since $\sup_{\|f\|_\infty = 1} |S_N f(0)| = \int_0^1 |D_N(t)|\,dt \to \infty$, it follows from the Uniform Boundedness Theorem, that there exists $f \in C(\mathbb{T})$ such that $\sup_N |S_N f(0)| = \infty$, i.e. the Fourier Series diverges at 0. $\square$