Math 8100 Final Exam

Wednesday the 17th of December 2014

The five problems on this exam have equal weighting.

1. Prove that if $f \in L^2([0,1])$ and
$$\int_0^1 f(x)\, x^n\, dx = 0$$
for all non-negative integers n, then $f(x) = 0$ for almost every $x \in [0,1]$.

2. Let μ be a Borel measure on $\mathbb{R}^n$ with the property that for any given $\varepsilon > 0$ and Borel set $E \subseteq \mathbb{R}^n$, there always exists an open set G such that $E \subseteq G$ and $\mu(G \setminus E) < \varepsilon$. Prove that
$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\}$$
for all Borel sets $E \subseteq \mathbb{R}^n$.

3. Let $(X, \mathcal{M})$ be a measurable space.
   (a) Prove that if μ is a measure on $(X, \mathcal{M})$ and $f : X \to \mathbb{R}$ is a function in $L^1(\mu)$, then
$$f \geq 0 \text{ almost everywhere} \iff \int_E f\, d\mu \geq 0 \text{ for all } E \in \mathcal{M}.$$
   (b) Let μ_1 and μ_2 be measures on $(X, \mathcal{M})$ and $f_1, f_2 : X \to \mathbb{R}$ functions in $L^1(\mu_1)$ and $L^1(\mu_2)$ respectively. Prove that if
$$\int_E f_1\, d\mu_1 = \int_E f_2\, d\mu_2 \quad \text{for all } E \in \mathcal{M},$$
then
$$\int_E |f_1|\, d\mu_1 = \int_E |f_2|\, d\mu_2 \quad \text{for all } E \in \mathcal{M}.$$

4. Suppose F is a closed set in $\mathbb{R}$, whose complement has finite Lebesgue measure, and let $\delta(x)$ denote the distance from x to F, namely $\delta(x) := \inf\{|x - y| : y \in F\}$.
   (a) Prove that δ satisfies the Lipschitz condition $|\delta(x) - \delta(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
   (b) Consider
$$I(x) := \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\, dy.$$
      i. Show that $I(x) = \infty$ for each $x \notin F$.
      ii. Show that $\int_F I(x)\, dx < \infty$ and conclude that $I(x) < \infty$ for almost every $x \in F$.

5. Consider the Hilbert space $L^2([0,\infty), \mu)$, with μ the Borel measure on $[0, \infty)$ defined by $d\mu = e^{-x}\, dx$.
   (a) Apply the Gram-Schmidt process to $1, x, x^2 \in L^2([0,\infty), \mu)$ to obtain the orthonormal polynomials
$$L_0(x) = 1, \quad L_1(x) = x - 1, \quad L_2(x) = (x^2 - 4x + 2)/2.$$
   Hint: First verify that $\int_0^\infty x^n\, d\mu = n!$ *for all non-negative integers* n.
   (b) Find
$$\max \int_0^\infty x^2 g(x)\, d\mu$$
   where g is subject to the restrictions
$$\int_0^\infty g(x)\, d\mu = \int_0^\infty x\, g(x)\, d\mu = 0 \quad \text{and} \quad \int_0^\infty |g(x)|^2\, d\mu = 1.$$