Math 8100 Exam 2

Wednesday the 3rd of December 2014

Answer any THREE of the following four problems

1. Let X be a measurable subset of $\mathbb{R}^n$ and $L^2(X)^*$ denote the *dual space* of $L^2(X)$, namely the space of all *continuous linear functional* on $L^2(X)$.
   (a) Let $g \in L^2(X)$. Prove that the mapping $\Lambda_g : L^2(X) \to \mathbb{C}$ defined by
   $$\Lambda_g(f) := \int_X f\overline{g}$$
   for each $f \in L^2(X)$ defines an element of $L^2(X)^*$ with operator norm $\|\Lambda_g\|_{L^2(X)^*} = \|g\|_{L^2(X)}$.
   (b) Let $\Lambda \in L^2(X)^*$.
      i. Verify that $M := \{f \in L^2(X) \,:\, \Lambda(f) = 0\}$ is a closed subspace of $L^2(X)$.
      ii. Prove that there exists a unique $g \in L^2(X)$ which represents Λ in the sense that $\Lambda(f) = \Lambda_g(f)$ for each $f \in L^2(X)$.
   Hint: You may use, without proof, the fact that $L^2(X) = M \oplus M^\perp$.

2. (a) i. Give the definition of the vector space $L^\infty(\mathbb{R}^n)$ and that of the corresponding norm $\|\cdot\|_\infty$.
      ii. Verify that $\|\cdot\|_\infty$ indeed defines a norm on the vector space $L^\infty(\mathbb{R}^n)$.
      iii. Carefully prove that $L^\infty(\mathbb{R}^n)$ when equipped with the norm $\|\cdot\|_\infty$ is in fact a Banach space.
   (b) Prove that
   $$L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$$
   and that for any measurable function $f : \mathbb{R}^n \to \mathbb{C}$ one in fact has
   $$\|f\|_2 \leq \|f\|_1^{1/2} \|f\|_\infty^{1/2}.$$

3. (a) Prove that if $f, g : \mathbb{R}^n \to \mathbb{C}$ are both measurable functions on $\mathbb{R}^n$, then
   $$h(x, y) := f(x - y)g(y)$$
   defines a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$.
   Hint: You may use, without proof, that $F(x, y) = f(x)$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$.
   (b) Prove that if $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then the *convolution* $f * g$, defined for each $x \in \mathbb{R}^n$ by
   $$f * g(x) := \int f(x - y)g(y)\, dy$$
   is also a well-defined function in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and moreover satisfies the following estimates:
      i. $\|f * g\|_\infty \leq \|g\|_1 \|f\|_\infty$
      ii. $\|f * g\|_1 \leq \|g\|_1 \|f\|_1$ [Carefully justify your use of Fubini/Tonelli]
      iii. $\|f * g\|_2 \leq \|g\|_1 \|f\|_2$ [Hint: First show that $|f * g|^2 \leq \|g\|_1 (|f|^2 * |g|)$]

4. (a) Let $\varphi \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(x)\, dx = 1$ and $\varphi_t(x) := t^{-1}\varphi(t^{-1}x)$. Prove that if $f : \mathbb{R} \to \mathbb{C}$ is bounded and uniformly continuous, then $f * \varphi_t$ converges uniformly to f as $t \to 0$.
   (b) Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove the *Weierstrass Approximation Theorem*, namely that for any $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \mathbb{R}$ such that
   $$\sup_{x \in [0,1]} |f(x) - P(x)| < \varepsilon.$$
   Hint: You may use, without proof, the fact that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$ and that this series converges uniformly on all compact subsets of $\mathbb{R}$.