Math 8100 Exam 1

Monday the 6th of October 2014

Answer any <u>FOUR</u> of the following five problems

1. (a) Let E be an arbitrary subset of $\mathbb{R}^n$.
   i. What is the definition of $m_*(E)$, the Lebesgue outer measure of E?
   ii. Prove that there exists a Borel set B with the property that $E \subseteq B$ and $m_*(B) = m_*(E)$.
   You can use, without proof, the fact that if $m_*(E) < \infty$ then for any given $\varepsilon > 0$ there exists an open set G with the property that $E \subseteq G$ and $m_*(G) < m_*(E) + \varepsilon$.

   (b) i. Recall that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if for any $\varepsilon > 0$ there exists an open set G with $E \subseteq G$ such that $m_*(G \setminus E) < \varepsilon$. Use this definition to prove the following characterization:
   A set $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if and only if there exists a Borel set B with the property that $E \subseteq B$ and $m_*(B \setminus E) = 0$.
   ii. Give an example (no proofs required) of a specific set $E \subseteq \mathbb{R}$ with $m_*(E) < \infty$ and a Borel set B with the property that $E \subseteq B$ but $m_*(B) - m_*(E) < m_*(B \setminus E)$.

2. (a) Give a definition of what it means to say that an extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Lebesgue measurable.

   (b) Prove, arguing directly from the definition you gave in Question 2a, that if $\{f_j\}_{j=1}^{\infty}$ is a sequence of extended real-valued Lebesgue measurable functions, then
   $$\liminf_{j\to\infty} f_j$$
   will also be a Lebesgue measurable function.

3. Recall that
   $$L^+(\mathbb{R}^n) := \{f : \mathbb{R}^n \to [0,\infty] \;:\; f \text{ is Lebesgue measurable}\}.$$
   (a) Let φ be a simple function in $L^+(\mathbb{R}^n)$, give the definition of the integral of φ and extend this definition to cover all functions $f \in L^+(\mathbb{R}^n)$.

   (b) Carefully state both Fatou's lemma and the Monotone Convergence Theorem for functions in L^+ and prove that each result implies the other.

4. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and $f : E \to \overline{\mathbb{R}}$ be an extended real-valued Lebesgue measurable function. We shall say that f is *almost bounded on E* if
   $$\textit{For any } \varepsilon > 0 \textit{ there exists } N > 0 \;\; \textit{such that} \;\; m(\{x \in E : |f(x)| \geq N\}) < \varepsilon.$$
   (a) Prove that
   $$f \text{ almost bounded on } E \quad \Longrightarrow \quad |f(x)| < \infty \text{ for almost every } x \in E$$
   (b) Give an example (no proofs required) showing that the converse to the statement above is false.

   (c) Prove that if $f \in L^1(E)$ or $m(E) < \infty$, then
   $$|f(x)| < \infty \text{ for almost every } x \in E \quad \Longrightarrow \quad f \text{ almost bounded on } E$$

5. (a) State any version of the Dominated Convergence Theorem.

   (b) Prove that $\{f_j\}_{j=1}^{\infty}$ is a sequence of functions in $L^1(\mathbb{R}^n)$ with the property that $\sum\limits_{j=1}^{\infty} \|f_j\|_1 < \infty$, then $\sum\limits_{j=1}^{\infty} f_j$ converges to an L^1 function and
   $$\int \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \int f_j. \qquad (1)$$
   You can use, without proof, the fact that (1) holds unconditionally for any sequence in L^+.