Theorem (Continuity in L^p)

If $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{y \to 0} \|\tau_y f - f\|_p = 0$$

where $\tau_y f(x) = f(x-y)$ for all $x \in \mathbb{R}^n$. In other words,
for each $y \in \mathbb{R}^n$

$$\int |f(x-y) - f(x)|^p \, dx \longrightarrow 0 \quad \text{as } y \to 0 .$$

Proof:

First we consider $g \in C_c(\mathbb{R}^n)$ and assume $|y| \le 1$.
Since g and $\tau_y g$ are supported in some common compact set K,
and g is uniformly continuous on K,

$$\int |\tau_y g - g|^p \le \|\tau_y g - g\|_\infty^p \, m(K) \longrightarrow 0 \quad \text{as } y \to 0 .$$

Now suppose $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$. We know $\exists\, g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \varepsilon/3$ (& by translation invariance $\|\tau_y f - \tau_y g\|_p < \varepsilon \ \forall y$).
Hence

$$\begin{aligned} \|\tau_y f - f\|_p &\le \|\tau_y (f-g)\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \\ &< \tfrac{2}{3}\varepsilon + \|\tau_y g - g\|_p , \end{aligned}$$

and $\|\tau_y g - g\|_p < \varepsilon/3$ if y is sufficiently small. $\square$