Two Applications of Minkowski's inequality for integrals

Theorem 1 (Special case of Young's inequality)

If $1 \le p \le \infty$ and $f \in L^p$, $g \in L^1$, then $f * g \in L^p$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1$$

Proof: Recall that $f*g(x) = \int f(x-y)g(y)\,dy$, thus

$$\|f*g\|_p = \left\| \int f(\cdot - y) g(y)\,dy \right\|_p$$

$$\overset{\text{Minkowski}}{\le} \int |g(y)| \cdot \|\tau_y f\|_p \,dy = \|f\|_p \|g\|_1 .$$

□

Theorem 2 (Approximation to the identity)

Suppose $\varphi \in L^1(\mathbb{R}^n)$ and $\int \varphi = 1$. Let $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}^n)$, or bounded and uniformly continuous if $p = \infty$, then

$$\lim_{t \to 0} \|f * \varphi_t - f\|_p = 0$$

i.e. $\quad f * \varphi_t \longrightarrow f$ in L^p as $t \to 0$.

[Recall that $\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)$ for all $x \in \mathbb{R}^n$].

Proofs of Theorem 2

Proof 1:

$$f*\varphi_t(x) - f(x) = \int [f(x-y) - f(x)]\, \varphi_t(y)\, dy \qquad \left(\text{using } \int \varphi_t = 1\right)$$

$$\overset{\text{let } y = tz}{=} \int [f(x-tz) - f(x)]\, \varphi(z)\, dz$$

Hence

$$\| f*\varphi_t - f\|_p \underset{\text{Minkowski}}{\leq} \int \underbrace{\| \tau_{tz} f - f\|_p}\, |\varphi(z)|\, dz$$

↳ bounded by $2\|f\|_p$ & $\to 0$ as $t \to 0$ for all fixed z. (Continuity in L^p)

Result follows by the dominated convergence theorem. □

Proof 2: Let $q = p/p-1$.

$$|f*\varphi_t(x) - f(x)| \leq \int |f(x-y) - f(x)|\, |\varphi_t(y)|^{1/p} |\varphi_t(y)|^{1/q}\, dy$$

$$\overset{\text{Hölder}}{\leq} \left(\int |f(x-y) - f(x)|^p |\varphi_t(y)|\, dy \right)^{1/p} \underbrace{\left(\int |\varphi_t(y)|\, dy \right)^{1/q}}_{= \|\varphi\|_1^{1/q}}.$$

Hence

$$\| f*\varphi_t - f\|_p^p \leq \|\varphi\|_1^{p/q} \int\int |f(x-y) - f(x)|^p |\varphi_t(y)|\, dy\, dx$$

$$\overset{\text{Tonelli}}{=} \|\varphi\|_1^{p/q} \int |\varphi_t(y)|\, \|\tau_y f - f\|_p^p\, dy.$$

We now use the fact:

(*) For any $\eta > 0$, $\displaystyle\int_{|y| \geq \eta} |\varphi_t(y)|\, dy \overset{y = tz}{=} \int_{|z| \geq \eta/t} |\varphi(z)|\, dz \to 0$ as $t \to 0$.

By "Continuity in L^p" we know that for any $\varepsilon>0$, $\underline{\exists\ \eta>0}$ such that if $|y|<\eta$, then $\|\tau_y f - f\|_p^p \leq \frac{\varepsilon}{2\|\varphi\|_1^p}$. We therefore write

$$\int |\varphi_t(y)|\, \|\tau_y f - f\|_p^p\, dy$$

$$= \underbrace{\int_{|y|\geq\eta} |\varphi_t(y)|\, \|\tau_y f - f\|_p^p\, dy}_{\leq 2^p \|f\|_p^p \int_{|y|\geq\eta} |\varphi_t(y)|\, dy} + \underbrace{\int_{|y|<\eta} |\varphi_t(y)|\, \|\tau_y f - f\|_p^p\, dy}_{\leq \underline{\frac{\varepsilon}{2\|\varphi\|_1^{p-1}}}}.$$

$$\leq \underline{\frac{\varepsilon}{2\|\varphi\|_1^{p-1}}} \text{ if } t \text{ is small enough, by } (*).$$

Hence for any $\varepsilon>0$,

$$\|f*\varphi_t - f\|_p \leq \|\varphi\|_1^{p/q} \int |\varphi_t(y)|\, \|\tau_y f - f\|_p^p\, dy$$

$$\leq \|\varphi\|_1^{p/q} \left(\frac{\varepsilon}{\|\varphi\|_1^{p-1}}\right)$$

$$= \varepsilon \qquad \underline{\text{provided}}\ t \text{ is sufficiently small.} \qquad \square.$$

__Corollary__ (of Theorem 2): $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ if $1\leq p<\infty$.

__Proof__: Same as for $p=1$.