Math 8100 Exam 1

Thursday 26th of September 2013

Answer any <u>FIVE</u> of the following six problems

1. Let $E \subseteq \mathbb{R}^n$. Recall that the definition of the Lebesgue outer measure of E, $m_*(E)$ is given by

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over all countable coverings $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.

   (a) i. Prove that any countable subset of $\mathbb{R}$ necessarily has zero outer measure.

   ii. Give an example (no proofs required) of an uncountable subset of $\mathbb{R}$ with zero outer measure.

   (b) Prove that for any $E \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ there exists an open set G with $E \subseteq G$ and

$$m_*(E) \leq m_*(G) \leq m_*(E) + \varepsilon.$$

   Be sure to prove both of the inequalities above.

   (c) Give an example (no proofs required) of *disjoint* subsets A and B of $\mathbb{R}$ for which

$$m_*(A \cup B) \neq m_*(A) + m_*(B).$$

2. (a) Let $E \subseteq \mathbb{R}^n$.

   i. Give a definition of what it means to say that E is Lebesgue measurable.

   ii. Prove that for any $\delta > 0$, the dilated set

$$\delta E := \{\delta x \, : \, x \in E\}$$

   is also Lebesgue measurable and satisfies $m(\delta E) = \delta^n m(E)$.

   (b) Let $E \subseteq \mathbb{R}^n$ be Lebesgue measurable and $f : E \to \overline{\mathbb{R}}$ be an extended real-valued function on E.

   i. Give a definition of what it means to say that f is a Lebesgue measurable function.

   ii. Prove that if $f : E \to \overline{\mathbb{R}}$ is Lebesgue measurable and $\delta > 0$, then $f_\delta : \delta E \to \overline{\mathbb{R}}$ defined by

$$f_\delta(x) := \delta^{-n} f(\delta^{-1} x)$$

   is also Lebesgue measurable.

   (c) Recall that

$$L^+(\mathbb{R}^n) := \{f : \mathbb{R}^n \to [0, \infty] \, : \, f \text{ is Lebesgue measurable}\}.$$

   i. Let φ be a simple function in $L^+(\mathbb{R}^n)$, give the definition of the integral of φ and extend this definition to cover all functions $f \in L^+(\mathbb{R}^n)$.

   ii. Let $E \subseteq \mathbb{R}^n$ be Lebesgue measurable. Prove that for any $f \in L^+(\mathbb{R}^n)$ and $\delta > 0$ we have

$$\int_E f(x)\, dx = \int_{\delta E} f_\delta(x)\, dx$$

   where $f_\delta(x) = \delta^{-n} f(\delta^{-1} x)$.

3. Let $\mathcal{M}(\mathbb{R}^n)$ denote the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$ and

$$L^+(\mathbb{R}^n) := \{f : \mathbb{R}^n \to [0,\infty] \,:\, f \text{ is Lebesgue measurable}\}.$$

For a given simple function φ in $L^+(\mathbb{R}^n)$ define $\mu : \mathcal{M}(\mathbb{R}^n) \to [0,\infty]$ by

$$\mu(A) := \int_A \varphi(x)\, dx.$$

(a) (Countably additivity of μ on $\mathcal{M}(\mathbb{R}^n)$)
Let $A_1, A_2, \ldots$ be a disjoint countable collection of sets in $\mathcal{M}(\mathbb{R}^n)$ and $A = \bigcup_{k=1}^{\infty} A_k$. Prove, using only the definition of the integral defining μ and the countable additivity of Lebesgue measure, that

$$\mu(A) = \sum_{k=1}^{\infty} \mu(A_k).$$

(b) (Continuity from below of μ)
Let $E_1, E_2, \ldots$ be a countable collection of sets in $\mathcal{M}(\mathbb{R}^n)$ that increases to E in the sense that $E_j \subseteq E_{j+1}$ for all j, and $E = \bigcup_{j=1}^{\infty} E_j$. Use the countable additivity of μ on $\mathcal{M}(\mathbb{R}^n)$ established in part (a) above to prove that

$$\mu(E) = \lim_{j\to\infty} \mu(E_j).$$

4. (a) State and prove the Monotone Convergence Theorem for functions in $L^+(\mathbb{R}^n)$.

You can use (without proof) the results from Question 3 above even if you did not attempt this question, but this is not the only approach. If you choose to take the approach of deducing the result in question from some other convergence result (such as Fatou's Lemma, the Dominated Convergence Theorem or Egorov's Theorem), then you will only receive full credit if you both carefully state and prove any additional results used.

(b) (Linearity of the integral on L^+) Prove that if f and g are in $L^+(\mathbb{R}^n)$, then

$$\int (f+g)(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Hint: You may assume the linearity of the integral when it is applied to simple functions.

5. Let $f : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function, in other words let $f \in L^+(\mathbb{R}^n)$.

(a) (Tchebychev's inequality)
Prove that

$$m\left(\{x \in \mathbb{R}^n \,:\, f(x) > \alpha\}\right) \le \frac{1}{\alpha} \int f(x)\, dx$$

for all $\alpha > 0$.

(b) Prove that

$$\int f(x)\, dx = 0 \quad \Longleftrightarrow \quad f = 0 \text{ almost everywhere.}$$

6. (a) State any version of the Dominated Convergence Theorem.

(b) Suppose that $f(x)$ and $xf(x)$ are both Lebesgue integrable functions on $\mathbb{R}$. Prove that the function

$$F(t) = \int_{\mathbb{R}} f(x) \cos(tx)\, dx.$$

is differentiable at every t and find a formula for $F'(t)$.