ROTH'S THEOREM

In this note we give a self-contained proof of the second most famous theorem of Klaus Roth, namely the very special case of Szemerédi's theorem (the density version of van der Waerden's theorem) where the arithmetic progressions are of length three.

Roth's Theorem. *Let $\delta > 0$. There exists an absolute constant $C > 0$ such that if $N \geq \exp\exp(C\delta^{-1})$ and $A \subset [1, N]$ with $|A| = \delta N$, then A necessarily contains a (non-trivial) arithmetic progression of length three.*

1. Fourier analysis on $\mathbb{Z}$

If $f : \mathbb{Z} \to \mathbb{C}$ and $\sum_{n\in\mathbb{Z}} |f(n)| < \infty$, then we define its **Fourier transform** by

$$\widehat{f}(\alpha) = \sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n\alpha}.$$

Our absolute summability assumption on f ensures that the infinite series defining $\widehat{f}$ converges uniformly and hence that $\widehat{f}$ is a continuous function on the circle. In this setting the Fourier inversion formula and Plancherel's identity are essentially immediate consequences of the familiar orthogonality relation

$$\int_0^1 e^{2\pi i n\alpha} d\alpha = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}.$$

Indeed it is an easy exercise, using the uniform convergence of the infinite series defining $\widehat{f}$, to then estalish:

(i) Fourier inversion formula

$$f(n) = \int_0^1 \widehat{f}(\alpha)e^{2\pi i n\alpha} d\alpha.$$

(ii) Plancherel's identity

$$\int_0^1 |\widehat{f}(\alpha)|^2 d\alpha = \sum_{n\in\mathbb{Z}} |f(n)|^2.$$

2. Proof of Roth's theorem

We introduce the trilinear form

$$\Lambda_3(f, g, h) := \sum_{n\in\mathbb{Z}}\sum_{d\in\mathbb{Z}} f(n)g(n + d)h(n + 2d) = \int_0^1 \widehat{f}(\alpha)\widehat{g}(-2\alpha)\widehat{h}(\alpha)\, d\alpha,$$

where the second identity can be easily verified using the Fourier inversion formula. The significance of the trilinear form is that $\Lambda_3(1_A, 1_A, 1_A)$ in fact equals the exact number of three-term arithmetic progressions in A (including the $|A|$ trivial progressions where $d = 0$). In order to prove Roth's theorem it therefore suffices to show that

$$\Lambda_3(1_A, 1_A, 1_A) > \delta N.$$

For technical reasons it shall be convenient to consider functions of mean value zero.

Definition 1 (Balanced function)**.** We define the *balanced* function of A to be

$$f_A = 1_A - \delta 1_{[1,N]}.$$

Since A has density δ on $[1, N]$ it is a simple exercise to verify that indeed $\sum f_A(n) = 0$. Writing (the second) 1_A as $1_A = \delta 1_{[1,N]} + f_A$ we obtain

$$\Lambda_3(1_A, 1_A, 1_A) = \delta \sum_{n\in\mathbb{Z}}\sum_{d\in\mathbb{Z}} 1_A(n)1_A(n + 2d) + \Lambda_3(1_A, f_A, 1_A), \tag{1}$$

and note (by considering the even and odd elements of A) that

$$\frac{|A|^2}{2} \leq \sum_{n\in\mathbb{Z}}\sum_{d\in\mathbb{Z}} 1_A(n)1_A(n + 2d) \leq |A|^2$$

The leading term in identity (1) is therefore approximately δ^3N^2 which is instructive as this is also approximately the number of three-term arithmetic progressions that we would expect A to contain if it where random, obtained by selecting each natural number from 1 to N independently with probability δ.

Definition 2 (ε-uniformity)**.** We say that A is *ε-uniform* if $\sup_{\alpha\in[0,1]} |\widehat{f_A}(\alpha)| \leq \varepsilon N$.

Lemma 3 (Uniform $\Rightarrow$ Quasirandom)**.** *If A is ε-*uniform *with $\varepsilon = \delta^2/4$, then $\Lambda_3(1_A, 1_A, 1_A) \geq \delta^3 N^2/4$.*

We note that that $\delta^3 N^2/4 > \delta N$ if $N \geq 8\delta^{-2}$ (say).

Proof of Lemma 3. We will show that under this regularity assumption on A the term $\Lambda(1_A, 1_A, f_A)$ is in fact an *error* term and satisfies the estimate

$$|\Lambda(1_A, f_A, 1_A)| \leq \delta^3 N^2/4.$$

This follows immediately from the Plancherel's identity since

$$|\Lambda(1_A, f_A, 1_A)| \leq \sup_{\alpha\in[0,1]} |\widehat{f_A}(\alpha)| \int_0^1 |\widehat{1_A}(\alpha)|^2 d\alpha \leq \varepsilon\delta N^2. \qquad \square$$

At the heart of our proof of Roth's theorem is the following result.

Lemma 4 (Non-uniform $\Rightarrow$ Additive structure)**.** *If A is not ε-uniform then there exists an arithmetic progression P with $|P| \geq \sqrt{\varepsilon N/64\pi}$ such that $|A \cap P| > (\delta + \varepsilon/8)|P|$.*

Proof of Lemma 4. We let $L = \sqrt{\varepsilon N/64\pi}$ and suppose that

$$|A \cap P| \leq (\delta + \varepsilon/8)|P|$$

for every arithmetic progression P, with $|P| \geq L$.

Let $\alpha \in [0,1]$ be arbitrary. It follows from Dirichlet's principle that there exists $q \leq 4\pi L$ such that $\|q\alpha\| \leq 1/4\pi L$. Defining

$$P_0 = \{\ell q \,:\, 1 \leq \ell \leq L\}$$

it is then easy to see that

$$|\widehat{1_{P_0}}(\alpha)| \geq L - \sum_{\ell=1}^{L} \big|e^{2\pi i \ell q\alpha} - 1\big| \geq L\big(1 - 2\pi L\|q\alpha\|\big) \geq L/2.$$

Since f_A has mean value zero it follows that

$$f_A * 1_{P_0}(n) = \sum_m f_A(m) 1_{P_0}(n-m)$$

also has mean value zero. Using the fact that $|g| = 2g_+ - g$ where $g_+ = \max\{g, 0\}$ denotes the *positive-part* function, it then follows that

$$\sum_{n\in\mathbb{Z}} \big(f_A * 1_{P_0}(n)\big)_+ = \frac{1}{2}\sum_{n\in\mathbb{Z}} |f_A * 1_{P_0}(n)| \geq \frac{1}{2}|\widehat{f_A}(\alpha)\widehat{1_{P_0}}(\alpha)| \geq \frac{L}{4}|\widehat{f_A}(\alpha)|.$$

However, by assumption

$$f_A * 1_{P_0}(n) = \sum_{\ell=1}^{L} f_A(n - \ell q) = |A \cap P_n| - \delta|P_n \cap [1,N]| \leq \varepsilon L/8$$

whenever $P_n := n - P_0 \subseteq [1,N]$. We note that if $A \cap P_n \neq \emptyset$ then $P_n \subseteq [1,N]$ for all but at most Lq values of n, and hence

$$\sum_{n\in\mathbb{Z}} \big(f_A * 1_{P_0}(n)\big)_+ \leq L(\varepsilon N/8 + 2Lq) \leq L\varepsilon N/4.$$

It therefore follows that $|\widehat{f_A}(\alpha)| \leq \varepsilon N$ for all $\alpha \in [0,1]$, and hence that A is ε-uniform. $\square$

Proof of Roth's Theorem. We assume that A contains no non-trivial 3-term arithmetic progressions. This will, for N large enough, lead us to a contradiction.

It follows from Lemmas 3 and 4 that if A contains no three-term arithmetic progression then there must exists a (long) arithmetic progression P_1 with $|P_1| = N_1 \geq \sqrt{\delta^2 N/256\pi}$ such that $|A \cap P_1| \geq (\delta + \delta^2/32)|P_1|$.

If we pass to this subprogression and rescale it to have common difference 1, we obtain a set $A_1 \subseteq [1, N_1]$ with $|A_1| = \delta_1 N_1$ where $N_1 \geq \delta N^{1/2}/\sqrt{256\pi}$ and $\delta_1 \geq \delta + \delta^2/32$ that still does not contain an arithmetic progression of length three. After iterating this argument $k = 64/\delta$ times the density increases beyond 1, that is $\delta_k > 1$, an absurdity if N_k also remains large. Since $N_k \geq \delta^2 N^{1/2^k}/256\pi$ it follows that $\log N_k \geq 2^{-k}\log N - c'\log\delta^{-1}$, for some $c' > 0$, and hence $N_k \gg 1$ whenever $\log N \geq e^{C/\delta}$ for some suitably large constant $C > 0$. $\square$