Math 4100/6100 **Fall 2017**

Exam 2

Math 4100: Answer any THREE of the following FIVE questions
Math 6100: Answer any FOUR of the following FIVE questions
** All questions are weighted equally.*

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$.
   i. Carefully state the ε-δ definition of what it means for f to be *continuous* at $x_0 \in \mathbb{R}$.
   ii. Carefully state the definition of f being *uniformly continuous* on some set $X \subseteq \mathbb{R}$.

   (b) Use the definition from (i) above to prove that $g(x) = \dfrac{x^2 + 2x - 5}{x - 2}$ is continuous at $x_0 = 1$.

   (c) Prove that $h(x) = x^{-1}$ is continuous on $(0, \infty)$, uniformly continuous on $[a, \infty)$ for any $a > 0$, but *not* uniformly continuous on $(0, \infty)$.

2. (a) Let f be a differentiable function on $[a, b]$.
   i. Prove that if f attains a minimum at some point $c \in (a, b)$, then $f'(c) = 0$.
   ii. Prove that if L lies between $f'(a)$ and $f'(b)$, then there exist $c \in (a, b)$ such that $f'(c) = L$.
   Hint: Consider $h(x) := f(x) - Lx$

   (b) Let g be a differentiable function on $[0, 2]$, with $g(0) = 0$ and $g(1) = g(2) = 1$.
   i. Show that $g'(c) = 1/2$ for some $c \in (0, 2)$.
   ii. Show that $g'(c) = 1/7$ for some $c \in (0, 2)$.

3. (a) Find the value of $f^{(17)}(0)$ if $f(x) = \dfrac{4x}{2 - x}$.

   (b) Give an example of an infinitely differentiable function that is <u>not</u> equal to its Taylor series.

   (c) i. Prove that if $h : [0, \infty) \to \mathbb{R}$ is twice differentiable on $[0, x]$, then

   $$h(x) = h(0) + h'(0)x + \frac{h''(c)}{2}x^2$$

   for some $c \in (0, x)$. *Hint: Apply the "Generalized MVT" to* $h(x) - h(0) - h'(0)x$ *and* x^2.
   ii. How well does $1 + x/2$ approximate $\sqrt{1 + x}$ on $[0, 1/10]$?

4. (a) i. Find the pointwise limit of

   $$f_n(x) = \frac{nx}{1 + nx}$$

   on $[0, \infty)$. Explain why we know that the convergence *cannot* be uniform on $[0, \infty)$.
   ii. Prove that the convergence *is* uniform on $[a, \infty)$ whenever $a > 0$.
   iii. Using the definition directly, argue that the convergence is *not* uniform on $(0, \infty)$.

   (b) Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of the rational numbers. For each $n \in \mathbb{N}$ define

   $$g_n(x) = \begin{cases} 1 \text{ if } x > r_n \\ 0 \text{ if } x \le r_n \end{cases} .$$

   Prove that

   $$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\, g_n(x)$$

   defines a strictly increasing function g on $\mathbb{R}$ that is continuous on $\mathbb{R} \setminus \mathbb{Q}$.

5. (a) Let $h_n(x) = (x^2 + 1/n)^{1/2}$.
   i. Prove that h_n is differentiable at 0 for each $n \in \mathbb{N}$.
   ii. Prove that h_n converges uniformly to $|x|$ for all $x \in \mathbb{R}$.

   (b) Show that

   $$h(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2 + x^n}$$

   defines a continuous function h on $[0, 1]$ that is differentiable on $[0, 1)$.