Math 4100/6100 **Fall 2017**

Review of Infinite Series

1. Important infinite series

Geometric series: $\sum_{n=0}^{\infty} r^n$ converges $\iff$ $|r|<1$. If $|r|<1$, then $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$.

The p-series: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges $\iff$ $p>1$.

2. Series Tests

Definition. Given a sequence $\{a_n\}$ let $s_n = a_1 + \cdots + a_n$ denote its nth partial sum, then

$$\{a_n\} \text{ summable} \iff \sum_{n=1}^{\infty} a_n \text{ converges} \iff \underline{\{s_n\} \text{ converges}}.$$

Theorem 1 (Cauchy Criterion)**.**

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \text{for every } \varepsilon>0, \text{ there exists } N\in\mathbb{N} \text{ such that } \left|\sum_{k=m+1}^{n} a_k\right| \le \varepsilon \text{ if } n>m\ge N.$$

Corollary 2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 3. *If $a_n \ge 0$ and $s_n = a_1 + \cdots + a_n$, then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \{s_n\} \text{ bounded.}$$

Theorem 4 (Cauchy Condensation Test)**.** *If $\{a_n\}$ is a decreasing sequence of non-negative terms, then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \text{ converges.}$$

* *In Math 3100 you may have instead learned the "Integral Test". These tests are commonly used to establish when the p-series-type sums converge. Theorem 4 is Theorem 3.27 in Rudin.*

Theorem 5 (Direct Comparison Test)**.** *If $|a_n| \le b_n$ for all sufficiently large $n\in\mathbb{N}$, then*

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Corollary 6 (Direct Comparison Test for Divergence)**.**

$$\text{If } 0 \le b_n \le a_n \text{ for all sufficiently large } n\in\mathbb{N} \text{ and } \sum_{n=1}^{\infty} b_n \text{ diverges, then } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Corollary 7 (Absolute Convergence implies Convergence)**.**

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Corollary 8 (Limit Comparison Test)**.** Suppose $a_n>0$, $b_n>0$, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L \ne 0$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges.}$$

Theorem 9 (Root Test)**.** *Let* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha$.

- *If* $\alpha < 1$, *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
- *If* $\alpha > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Recall, by considering for example $\sum \frac{1}{n}$ *and* $\sum \frac{1}{n^2}$, *that the Root Test is inconclusive if* $\alpha = 1$.

Theorem 10 (Ratio Test)**.** *Let* $\{a_n\}$ *be a sequence of non-zero terms.*

- *If* $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
- *If there exists an* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq N$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

The Ratio Test is also inconclusive if either $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ *or* $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Although Theorem 10 can easily be established directly, it also follows from Theorem 9 and the following

Lemma 11. *If* $\{c_n\}$ *is any sequence of positive real numbers, then*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Partial Summation: If $s_n = a_1 + \cdots + a_n$, then it is easy to verify that

$$\sum_{k=m+1}^{n} a_k b_k = s_n b_{n+1} - s_m b_{m+1} + \sum_{k=m+1}^{n} s_k (b_k - b_{k+1}).$$

If we let $n \to \infty$ we see that the series $\sum a_k b_k$ converges if both the series $\sum s_k(b_k - b_{k+1})$ and the sequence $\{s_n b_{n+1}\}$ converge, the next two tests give sufficient conditions for this to indeed happen.

Theorem 12 (Dirichlet Test)**.**

$$\{s_n\} \text{ bounded and } \{b_n\} \text{ decreasing with limit } 0 \implies \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

Corollary 13 (Alternating Series Test)**.** If $\{b_n\}$ is decreasing with limit 0, then $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Theorem 14 (Abel Test)**.**

$$\sum_{n=1}^{\infty} a_n \text{ convergent and } \{b_n\} \text{ monotone and bounded} \implies \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

3. Strategy for analyzing $\sum_{n=1}^{\infty} a_n$

1. Does $a_n \to 0$?

   If NO, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Does $\sum_{n=1}^{\infty} |a_n|$ converge?

   If YES, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and hence converges. Try using

   - geometric series and p-series
   - first and second comparison tests
   - ratio and root tests
   - Cauchy condensation test (or integral test)

3. If $\sum_{n=1}^{\infty} |a_n|$ does not converge or you cannot decide, then try

   - alternating series test
   - partial summation (Dirichlet or Abel Test)

   If these tests apply, then $\sum_{n=1}^{\infty} a_n$ converges.

   Recall that if

   $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ converges conditionally.