Math 4100/6100 Assignment 7
More on Continuity

Due date: By 12:00 pm on Friday the 20th of October 2017

1. (a) i. Show that $f(x) = x^3$ is continuous on all of $\mathbb{R}$.
   ii. Argue that f is however not uniformly continuous on $\mathbb{R}$.
   (b) Show that $g(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$, but not on the set $(0, 1]$.
   (c) Show that $h(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

2. Give an example of each of the following, or provide a short argument for why the request is impossible.
   (a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.
   (b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.
   (c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

3. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be continuous. Prove that

$$Z(f) := \{x \in X \,:\, f(x) = 0\}$$

   is a closed subset of X.

4. Let (X, d) be a metric space.
   (a) If E is a non-empty subset of X, we define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

   i. Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
   ii. Prove that ρ_E is uniformly continuous on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

   for all $x, y \in X$.
   (b) Let A and B be disjoint non-empty closed subsets of X, and define

$$f(x) := \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

   for each $x \in X$. Show that f is a continuous function from X into $[0, 1]$ such that $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.
   Note that this in particular establishes a converse to Q3: Every closed set in X is $Z(f)$ for some continuous function $f : X \to \mathbb{R}$. It also gives another proof that $\mathbb{R}$ cannot be written as a disjoint union of two non-empty closed subsets.

Math 6100/Bonus Problems

1. Suppose K and F are disjoint sets in $\mathbb{R}$, with K compact, and F closed.
   (a) Prove that there exist $\delta > 0$ such that $|x - y| > \delta$ whenever $x \in K$ and $y \in F$.
   Hint: Use the fact, see Q4(a)i. above, that ρ_F is a continuous positive function on K.
   (b) Show that there in fact exist points $x_0 \in K$ and $y_0 \in F$ such that

$$|x_0 - y_0| = \inf\{|x - y| \,:\, x \in K \text{ and } y \in F\}.$$

   (c) Show that it is possible to have $\inf\{|x - y| \,:\, x \in F_1 \text{ and } y \in F_2\} = 0$ for two disjoint closed sets F_1 and F_2 if neither is assumed to be compact.

Challenge Problem

1. Construct a function $f : \mathbb{R} \to \mathbb{R}$ with the *truly amazing* property that the image of every open interval is the whole of $\mathbb{R}$.