Math 4100/6100 Assignment 5
Upper and Lower Limits and More Basic Topology

Due date: 12:00 pm on Friday the 29th of September 2017

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.
   (a) Prove that $\{a_n\}$ is unbounded above if and only if it has a subsequence with limit $+\infty$.
   (b) Prove that $\{a_n\}$ is unbounded below if and only if it has a subsequence with limit $-\infty$.

2. Let $\{x_n\}$ be a bounded sequence. Prove statements (a) and (b) below directly <u>twice</u>, once each using the following equivalent definitions:
   (i) $\limsup_{n\to\infty} x_n := \sup\{x \in \mathbb{R} : x \text{ is a subsequential limit of } \{x_n\}\}$
   (ii) $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k$

   (a) If $|x_n| \leq M$ for all $n \in \mathbb{N}$, then $|\limsup_{n\to\infty} x_n| \leq M$ also.
   (b) If $\beta > \limsup_{n\to\infty} x_n$, then there exists a $N \in \mathbb{N}$ such that $x_n < \beta$ for all $n \geq N$.

3. (a) Let $\{x_n\}$ be a bounded sequence. Prove that if $\limsup_{n\to\infty} |x_n| = 0$, then $\lim_{n\to\infty} x_n$ exists and equals 0.
   (b) Prove that a bounded sequence that does not converge always has at least two subsequences that converge to different limits.
   (c) Find the limit inferior and limit superior of the sequence $\{a_n\}$ if $a_n = \lfloor \sin n \rfloor$ for all $n \in \mathbb{N}$.
   (d) Find the set of all subsequential limits for the sequence $\{x_n\}$ if for all $n \in \mathbb{N}$

   (i) $x_n = 4 + 5(-1)^{\lfloor n/2 \rfloor}$ (ii) $x_n = \cos(n\pi/3)$ (iii) $x_n = (-1)^{\lfloor n/2 \rfloor} + 2(-1)^{\lfloor n/3 \rfloor}$

4. (a) Explain why there is no sequence whose set of subsequential limits is $\{1/n : n \in \mathbb{N}\}$.
   (b) Give an example of a sequence whose set of subsequential limits is $\{1/n : n \in \mathbb{N}\} \cup \{0\}$.

5. For any two bounded sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

6. Prove that if $\{G_1, G_2, \ldots\}$ is a countable collection of dense, open subsets of $\mathbb{R}$, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty. Prove that this intersection is in fact dense in $\mathbb{R}$.
   Hint: Imitate the proof that perfect subsets in $\mathbb{R}$ are uncountable – I will help you with this in class!

7. Prove that $\mathcal{C} + \mathcal{C} = [0, 2]$, where $\mathcal{C}$ denotes the usual (middle-third) Cantor set and

$$\mathcal{C} + \mathcal{C} = \{x + y : x, y \in \mathcal{C}\}.$$

   Hint: Consider the intersection of the set $\mathcal{C} \times \mathcal{C} \subset \mathbb{R}^2$ and the family of lines $\{x + y = c \,|\, c \in [0, 2]\}$ and use the property of nested compact sets.

Math 6100/Bonus Problems

1. Prove that every open set in $\mathbb{R}$ is the union of at most a countable collection of disjoint open intervals.
2. Prove that $\mathbb{R}$ cannot be written as the disjoint union of two non-empty closed sets.
3. Construct a bijection from $\mathbb{R}$ to its proper subset $\mathbb{R} \setminus \mathbb{Q}$ of irrationals.