Math 4100/6100 Assignment 3 & 4
Basic Topology

Due date: 5:00 pm on Thursday the 21st of September 2017

* In Questions 1-6 below all sets are assumed to be in $\mathbb{R}$ with $\mathbb{R}$ equipped with its usual Euclidean metric.

1. Let
$$E = \left\{ \frac{(-1)^n n}{n+1} \, : \, n \in \mathbb{N} \right\}.$$
   (a) Find the limit points of E.
   (b) Is E a closed set? Is E an open set?
   (c) Does E contain any isolated points?
   (d) Find $\overline{E}$, the closure of E.

2. Construct a bounded set of real numbers with exactly three limit points.

3. Decide which of the following subsets of $\mathbb{R}$ are open, closed, or neither (with respect to the usual metric on $\mathbb{R}$). If the set is not open, find a point in the set for which there is no ε-neighborhood contained in the set. If the set is not closed, find a limit point that is not contained in the set.
   (a) $\mathbb{Q}$
   (b) $\mathbb{N}$
   (c) $(0, \infty)$
   (d) $(0, 1]$
   (e) $\{1 + 1/4 + \cdots + 1/n^2 \, : \, n \in \mathbb{N}\}$

4. Decide whether the following sets are compact. For those which are not compact, show how the definitions of both sequentially compact and compact break down. In other words, give an example of (i) a sequence contained in the set that does not possess a subsequence converging to a limit in the set, and (ii) an open cover for which there is no finite subcover.
   (a) $\mathbb{Q}$
   (b) $\mathbb{Q} \cap [0, 1]$
   (c) $\mathbb{R}$
   (d) $\mathbb{Z} \cap [0, 10]$
   (e) $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$
   (f) $\{1, 1/2, 2/3, 3/4, 4/5, \dots\}$

5. Let $E \subseteq \mathbb{R}$ consist of 0 and the numbers $1/n$ with $n \in \mathbb{N}$. Prove that E is compact directly from the definition.

6. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.
   (a) For any set $E \subseteq \mathbb{R}$, $\overline{E}^c$ is open.
   (b) If a set has an isolated point, it cannot be an open set.
   (c) If E is a bounded set, then $s = \sup E$ is a limit point of E.
   (d) Every non-empty compact subset of $\mathbb{R}$ has a largest member.
   (e) An open set in $\mathbb{R}$ that contains every rational number must be all of $\mathbb{R}$
   (f) An arbitrary intersection of compact sets is compact.

(g) Let $E \subseteq \mathbb{R}$ be arbitrary, and let $K \subseteq \mathbb{R}$ be compact. Then the intersection $E \cap K$ is compact.

(h) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a nested sequence of non-empty closed sets, then the intersection

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

(i) A finite set is always compact.

(j) A countable set is always compact.

7. Let (X, d) be a metric space and $E \subset X$.
   Prove that if $x = \lim_{n\to\infty} x_n$ for some sequence $\{x_n\}$ in E with $x \cap \{x_n\} = \emptyset$, then x is a limit point of E.

8. Let (X, d) be a metric space, $E \subset X$ and E' denote the set of all limit points of a set E.

   (a) Prove that E' is closed.

   (b) Prove that if $x \in (A \cup B)'$, then either $x \in A'$ or $x \in B'$.

   (c) Use parts (a) and (b) to give a new proof of the fact that the closure of E, $\overline{E} = E \cup E'$, is closed.

9. Let X be a infinite set. For $x, y \in X$, define

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

   (a) Prove that this is a metric.

   (b) Which subsets of the resulting metric space are open? Which are closed? Which are compact?

   (c) Show that in this setting $\overline{B_\varepsilon(x)} \neq \{y \in X \,:\, d(x,y) \leq \varepsilon\}$ in general.

10. Determine which of the following are metrics on $\mathbb{R}$.

   (a) $d_1(x,y) = (x-y)^2$

   (b) $d_2(x,y) = \sqrt{|x-y|}$

   (c) $d_3(x,y) = |x^2 - y^2|$

   (d) $d_4(x,y) = |x - 2y|$

   (e) $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$

Math 6100/Bonus Problems

1. Show directly that compact sets are always both closed and bounded (without using the notion of sequential compactness).

2. Construct a compact set of real numbers whose limit points form a countable set.

3. Let $E \subseteq \mathbb{R}$ be uncountable and E' denote the set of limit points of E. Prove that $E \cap E'$ is uncountable.

4. Let $\mathcal{B}$ denote the set of all *Bernoulli sequences*, i.e., sequences $\{x_n\}$ with $x_n \in \{0,1\}$ for all $n \in \mathbb{N}$.

   (a) Prove that $\rho((x_n),(y_n)) = \sum_{n=0}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on $\mathcal{B}$.

   (b) Prove that the set of all sequences in $\mathcal{B}$ which begin $0, 1$ (in that order) is both *open* and *closed*.

Challenge Problems

1. Prove that $\mathbb{R}$ cannot be written as the disjoint union of two non-empty closed sets.

2. Construct a bijection from $\mathbb{R}$ to its proper subset $\mathbb{R} \setminus \mathbb{Q}$ of irrationals.