Math 4100/6100 Assignment 2
Cardinality of Sets

Due date: 12:00 pm on Friday the 1st of September 2017

1. (a) Give an example of a countable collection of disjoint open intervals in $\mathbb{R}$.

   (b) Give an example of an uncountable collection of disjoint open intervals in $\mathbb{R}$, or argue that no such collection exists.

2. Let A be a countable set, and A_n denote the collection of all n-tuples $(a_1, \ldots, a_n)$ with each $a_j \in A$ for $1 \leq j \leq n$ (these elements need not be distinct). Prove that A_n is countable for each $n \in \mathbb{N}$.

3. A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

   Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers.

   (a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2} + \sqrt{3}$ are algebraic.

   (b) Prove that the set of all algebraic numbers is countable. What may we conclude from this regarding the set of all transcendental numbers?

   Hint: First show that the set of all polynomials with integer coefficients of degree n is countable.

4. (a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

   (b) Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ uncountable?

   (c) Does the statement in (a) remain true if "uncountable" is replaced with "infinite"?

5. (a) Let A be a given set and $P(A)$ denote the *power set* of A, namely the collection of all subsets of A. Prove that there does not exist a function $f : A \to P(A)$ that is onto.

   Hint: Assume that such a function does exist and arrive at a contradiction by considering the set

$$B = \{a \in A : a \notin f(a)\}.$$

   (b) Prove that the *set* of all infinite subsets of $\mathbb{N}$ is uncountable.

   Hint: Show directly that the set of all finite subsets of $\mathbb{N}$ is countable.

Math 6100/Bonus Problems

1. (Schröder–Bernstein Theorem). Assume there exists a 1–1 function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Follow the steps to show that there exists a 1–1, onto function $h : X \to Y$ and hence $X \sim Y$.

   The strategy is to partition X and Y into components $X = A \cup A'$ and $Y = B \cup B'$ with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

   (a) Explain how achieving this would lead to a proof that $X \sim Y$.

   (b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.

   (c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

   (d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show that g maps B' onto A'.