Math 4100/6100 Assignment 10
Integration

Not to be handed in for grading

1. Let f be a bounded function on $[a, b]$.
   (a) Show that $U(f) \geq L(f, P)$ for any partition P of $[a, b]$.
   (b) Prove that $U(f) \geq L(f)$.

2. Let f be a bounded real-valued function on $[a, b]$. Prove that f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$ for every choice of $c \in (a, b)$ and that in this case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

3. (a) Prove that if $f_n \to f$ uniform on $[a, b]$ and each f_n is integrable, then f is also integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

   (b) Give an examples showing that if the convergence is merely pointwise, then f need not necessarily be integrable and even if it is final conclusion does not necessarily hold.

4. Provide an example or give a reason why the request is impossible.
   (a) A sequence $\{f_n\}$ that converges pointwse to f where each f_n has at most a finite number of discontinuities but f is not integrable.
   (b) A sequence $\{g_n\}$ that converges uniformly to g where each g_n has at most a finite number of discontinuities and g is not integrable.
   (c) A sequence $\{h_n\}$ that converges uniformly to h where each h_n is not integrable but h is integrable.

5. Let $\{r_n\}$ be any enumeration of all the rationals in $[0, 1]$ and define $f : [0, 1] \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}.$$

   Prove, directly from the definition, that f is integrable on $[0, 1]$ and $\int_0^1 f(x)\,dx = 0$.

6. Recall *Thomae's function* $g : \mathbb{R} \to \mathbb{R}$, by

$$g(x) := \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{1}{n} & \text{if } x = \dfrac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   Prove, directly from the definition, that g is integrable on $[0, 1]$ and $\int_0^1 g(x)\,dx = 0$.

7. Let $\mathcal{C}$ denote the usual "middle third" Cantor set and

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{if } x \notin \mathcal{C} \end{cases}.$$

   Prove, directly from the definition, that h is integrable on $[0, 1]$ and $\int_0^1 h(x)\,dx = 0$.

8. Decide which of the following conjectures is true and supply a short proof. For those which are not true, give a counterexample.

   (a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on $[a, b]$.
   (b) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g(x)\, dx > 0$.
   (c) If h is continuous on $[a, b]$ and $h(x) \geq 0$ on $[a, b]$ with $h(x_0) > 0$ for at least one point $x_0 \in [a, b]$, then $\int_a^b h(x)\, dx > 0$.

9. Show that if $f(x) > 0$ for all $x \in [a, b]$ and f is integrable on $[a, b]$, then $\int_a^b f(x)\, dx > 0$

10. Decide whether the each statement is true or false, providing a short justification for each conclusion.

    (a) If $h' = g$ on $[a, b]$, then g is continuous on $[a, b]$.
    (b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.
    (c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c.

11. Let $g : [a, b] \to \mathbb{R}$ be integrable.

    (a) Prove that the function $G(x) = \displaystyle\int_a^x g(t)\, dt$ is uniformly continuous on $[a, b]$.
    (b) Prove that

    $$\int_a^c g(t)\, dt = \int_c^b g(t)\, dt$$

    for some $c \in [a, b]$. Give an example to show that c may be one of the endpoints.

12. Prove that if f and g are two continuous functions on $[a, b]$ and $g(x) \geq 0$ for all $x \in [a, b]$, then there must exist a point $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\, dx = f(c) \cdot \int_a^b g(x)\, dx.$$