Math 4100/6100 Assignment 1

Some Preliminaries and a Review of Sequences and Series

Due date: 5:00pm on Thursday the 24th of August 2017

1. (Reverse Triangle Inequality). Use the triangle inequality to prove that if $x, y \in \mathbb{R}$, then
$$\big||x| - |y|\big| \leq |x - y|.$$

2. (a) Let $q \neq 0$ be rational and x be irrational. Prove that $q + x$ and qx are both irrational.
   (b) Use the *Archimedean Property* of $\mathbb{R}$ to prove that between any two distinct real numbers there is both a rational and irrational number.

3. (a) (De Morgan's Laws). Let A and B be subsets of $\mathbb{R}$. Verify the following:
      i. $(A \cap B)^c = A^c \cup B^c$
      ii. $(A \cup B)^c = A^c \cap B^c$
   (b) i. Show how induction can be used to conclude that
   $$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$
   for any finite $n \in \mathbb{N}$.
      ii. Explain why induction *cannot* be used to conclude that
   $$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c.$$
      iii. Is the statement in part (ii) above valid? Give either a proof or counterexample.

4. (a) Let $A \subseteq \mathbb{R}$ be non-empty and bounded below. Show that
      i. $\inf A = -\sup(-A)$ where $-A = \{-x \,:\, x \in A\}$
      ii. $\inf A = \sup(B)$ where $B = \{b \,:\, b \text{ is a lower bound for } A\}$
   (b) Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.
      i. Show that if $A \subseteq B$, then $\sup A \leq \sup B$.
      ii. Show that if $\sup A < \sup B$, then there must exist a $b \in B$ that is an upper bound for A.

5. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.
   (a) $\displaystyle\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$ (b) $\displaystyle\lim_{n\to\infty} \frac{1}{6n^2+1} = 0$ (c) $\displaystyle\lim_{n\to\infty} \frac{2}{\sqrt{n+3}} = 0$

6. What happens if we reverse the order of the quantifiers in the definition of convergence of a sequence?

   Definition: A sequence $\{a_n\}$ *verconges* to a if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|a_n - a| < \varepsilon$.

   Give an example of a vercongent sequence. Can you give an example a vercongent sequence that is divergent? What exactly is being described in this strange definition?

7. Verify the following using the definition of convergence of a sequence:
   (a) If $a_n \to a$, then $|a_n| \to |a|$. Is the converse true?
   (b) If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \to a$, then $\sqrt{a_n} \to \sqrt{a}$.
   (c) If $\{a_n\}$ is a bounded but not necessarily convergent sequence and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
   (d) If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$ as well.
   Note that this immediately (right?) implies the following "Squeeze Theorem":

   If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

8. Let $\{a_n\}$ be a convergent sequence with $\lim_{n\to\infty} a_n = a$. Prove the following two statements:
   (a) If $a_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq b$.
   (b) If $\{a_n\}$ is increasing, then $a_n \leq a$ for all $n \in \mathbb{N}$.

9. Let $a_1 = \sqrt{2}$, and define $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$. Prove that $\lim_{n\to\infty} a_n$ exists and equals 2.

10. (a) Investigate the behavior (convergence or divergence) of $\sum_{n=1}^{\infty} a_n$ if

    (i) $a_n = \sqrt{n+1} - \sqrt{n}$ (ii) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$ (iii) $a_n = \left(\sqrt[n]{n} - 1\right)^n$.

    (b) Let $a_n > 0$ for all $n \in \mathbb{N}$.
      i. Show that in $\lim_{n\to\infty} n a_n$ exists and is not equal to 0, then $\sum_{n=1}^{\infty} a_n$ diverges.
      ii. Show that in $\lim_{n\to\infty} n^2 a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.

    (c) Prove that if $a_n > 0$ for all $n \in \mathbb{N}$, then the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of both

    (i) $\sum_{n=1}^{\infty} a_n^2$ (ii) $\sum_{n=1}^{\infty} \dfrac{\sqrt{a_n}}{n}$.

Math 6100/Bonus Problems

1. Suppose $\{a_n\}$ is a sequence of real numbers and $b_n = \dfrac{a_1 + \cdots + a_n}{n}$.
   Prove that if $a_n \to 0$, then $b_n \to 0$. Is the converse true? What if $a_n \to L$?

2. (a) Use Question 8 to deduce the *Nested Interval Property* from the *Monotone Convergence Theorem.*
   (b) Show conversely that one can also deduce the *Monotone Convergence Theorem* from the *Nested Interval Property.*

3. Directly show the equivalence of the *Bolzano-Weierstrass Theorem* and the *Nested Interval Property.*