Math 4100/6100 Assignment 4

Due date: 5:00 pm on Thursday the 22nd of September 2016

Basic Warm-up Problems (not to be handed in with the assignment)

1. Let $\{a_n\}$ be a convergent sequence with $\lim_{n\to\infty} a_n = a$. Prove the following two statements:
   (a) If $a_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq b$.
   (b) If $\{a_n\}$ is increasing, then $a_n \leq a$ for all $n \in \mathbb{N}$.

2. (a) Prove that if $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ as well.
   (b) Prove that the convergence of $\{a_n\}$ implies the convergence of $\{|a_n|\}$. Is the converse true?

Sequential and Subsequential Limits

1. What happens if we reverse the order of the quantifiers in the definition of convergence of a sequence?

   Definition: A sequence $\{a_n\}$ *verconges* to a if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|a_n - a| < \varepsilon$.

   Give an example of a vercongent sequence. Can you give an example a vercongent sequence that is divergent? What exactly is being described in this strange definition?

2. Here are two slightly non-standard definitions that we discussed in class:
   (i) A sequence $\{a_n\}$ is *eventually* in a set $V \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in V$ for all $n \geq N$.
   (ii) A sequence $\{a_n\}$ is *frequently* in a set $V \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in V$.

   (a) Is the sequence $\{(-1)^n\}$ eventually or frequently in the set $\{1\}$?
   (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?
   (c) Suppose an infinite number of terms of a sequence $\{a_n\}$ are equal to 2. Is $\{a_n\}$ necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

3. (a) Show that the *Cauchy Criterion* implies the *Monotone Convergence Theorem.*
   (b) Show that the *Monotone Convergence Theorem* implies the *Nested Interval Property.*
   (c) Show that the *Nested Interval Property* implies the *Axiom of Completeness.*

4. Let $\{x_n\}$ be a bounded sequence. Prove statements (a) and (b) below directly <u>twice</u>, once each using the following equivalent definitions:
   (i) $\limsup_{n\to\infty} x_n := \sup \{x \in \mathbb{R} : x \text{ is a subsequential limit of } \{x_n\}\}$
   (ii) $\limsup_{n\to\infty} x_n := \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k$

   (a) If $|x_n| \leq M$ for all $n \in \mathbb{N}$, then $|\limsup_{n\to\infty} x_n| \leq M$ also.
   (b) If $\beta > \limsup_{n\to\infty} x_n$, then there exists a $N \in \mathbb{N}$ such that $x_n < \beta$ for all $n \geq N$.

5. (a) Let $\{x_n\}$ be a bounded sequence. Prove that if $\limsup_{n\to\infty} |x_n| = 0$, then $\lim_{n\to\infty} x_n$ exists and equals 0.
   (b) Prove that a bounded sequence that does not converge always has at least two subsequences that converge to different limits.
   (c) Find the limit inferior and limit superior of the sequence $\{a_n\}$ if $a_n = \lfloor \sin n \rfloor$ for all $n \in \mathbb{N}$.
   (d) Find the set of all subsequential limits for the sequence $\{x_n\}$ if for all $n \in \mathbb{N}$

   (i) $x_n = 4 + 5(-1)^{\lfloor n/2 \rfloor}$ (ii) $x_n = \cos(n\pi/3)$ (iii) $x_n = (-1)^{\lfloor n/2 \rfloor} + 2(-1)^{\lfloor n/3 \rfloor}$

6. (a) Explain why there is no sequence whose set of subsequential limits is $\{1/n \ : n \in \mathbb{N}\}$.
   (b) Give an example of a sequence whose set of subsequential limits is $\{1/n \ : n \in \mathbb{N}\} \cup \{0\}$.

7. Find the upper and lower limits, namely $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$, of the sequence $\{a_n\}$ defined by

$$a_1 = 0; \qquad a_{2m} = \frac{a_{2m-1}}{2}; \qquad a_{2m+1} = \frac{1}{2} + a_{2m}.$$

8. For any two bounded sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

9. (a) Let $\{a_n\}$ denote a bounded sequence of positive reals. Prove that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$

   (b) Can you define a sequnence $\{a_n\}$ for which the inequalities above are all strict?
   (c) Use the result in part (a) above to prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

More Basic Topology of $\mathbb{R}$

1. Prove that if $\{G_1, G_2, \dots\}$ is a countable collection of dense, open subsets of $\mathbb{R}$, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty. Prove that this intersection is in fact dense in $\mathbb{R}$.
   Hint: Imitate the proof that perfect subsets in $\mathbb{R}$ are uncountable – I will help you with this in class!

2. This question deals with the G_δ and F_σ subsets of $\mathbb{R}$ that have been discussed in lecture, see also Definition 3.5.1 in Abbott.
   (a) Show that every closed set is a G_δ set and every open set is an F_σ set.
   Hint: If F is closed, consider $O_n = \{x : \inf_{y\in F} |x - y| < 1/n\}$.
   (b) Give an example of an F_σ set which is not a G_δ set.
   Hint: Use Question 1.
   (c) Give an example of a set which is neither an F_σ nor a G_δ set.

Math 6100/Bonus Problems

1. Prove that every open set in $\mathbb{R}$ is the union of at most a countable collection of disjoint open intervals.

2. Construct a compact set of real numbers whose limit points form a countable set.