Math 4100/6100 Assignment 1

Some Preliminaries and Cardinality of Sets

Due date: 5:00pm on Tuesday the 23rd of August 2016

Warm-up Problems

1. Let $q \neq 0$ be rational and x be irrational. Prove that $q + x$ and qx are both irrational.

2. Form the logical negation of each of the following claims (all of which we actually know to be true, right?). One way to do this is to simply add "It is not the case that..." in front of each assertion, but for each statement, try and embed the word "not" as deeply into the resulting sentence as possible (or avoid using it altogether).

   (a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.

   (b) Between every two distinct real numbers, there is an irrational number.

   (c) Given any real number x, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

3. Use the first part of Q1 above together with the fact, proved in class, that between any two distinct real numbers there is a rational to deduce the validity of statement (b) in Q2 above.

Preliminaries about subsets of $\mathbb{R}$

1. (De Morgan's Laws). Let A and B be subsets of $\mathbb{R}$. Verify the following:

   (a) $(A \cap B)^c = A^c \cup B^c$

   (b) $(A \cup B)^c = A^c \cap B^c$

2. For this exercise, assume that Question 1 has been successfully completed.

   (a) Show how induction can be used to conclude that

   $$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

   for any finite $n \in \mathbb{N}$.

   (b) Explain why induction *cannot* be used to conclude that

   $$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c.$$

   (c) Is the statement in part (b) above valid? If so, write a proof that does not use induction.

3. Let $A \subseteq \mathbb{R}$ be non-empty and bounded below. Show that

   (a) $\inf A = -\sup(-A)$ where $-A = \{-x : x \in A\}$

   (b) $\inf A = \sup(B)$ where $B = \{b : b \text{ is a lower bound for } A\}$

4. Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.

   (a) Show that if $A \subseteq B$, then $\sup A \leq \sup B$.

   (b) Show that if $\sup A < \sup B$, then there exists a $b \in B$ that is an upper bound for A.

Cardinality of Sets

1. (a) Give an example of a countable collection of disjoint open intervals.

   (b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

2. Let A be a countable set, and A_n denote the collection of all n-tuples $(a_1, \dots, a_n)$ with each $a_j \in A$ for $1 \leq j \leq n$ (these elements need not be distinct). Prove that A_n is countable for each $n \in \mathbb{N}$.

3. A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, \dots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

   Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers.

   (a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2} + \sqrt{3}$ are algebraic.

   (b) Prove that the set of all algebraic numbers is countable. What may we conclude from this regarding the set of all transcendental numbers?

   Hint: First show that the set of all polynomials with integer coefficients of degree n is countable.

4. (a) Let A be a given set and $P(A)$ denote the *power set* of A, namely the collection of all subsets of A. Prove that there does not exist a function $f : A \to P(A)$ that is onto.

   Hint: Assume that such a function does exist and arrive at a contradiction by considering the set

$$B = \{a \in A \,:\, a \notin f(a)\}.$$

   (b) Prove that the *set* of all infinite subsets of $\mathbb{N}$ is uncountable.

   Hint: Show directly that the set of all finite subsets of $\mathbb{N}$ is countable.

Math 6100/Bonus Problems

1. (Schröder–Bernstein Theorem). Assume there exists a 1–1 function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Follow the steps to show that there exists a 1–1, onto function $h : X \to Y$ and hence $X \sim Y$.

   The strategy is to partition X and Y into components $X = A \cup A'$ and $Y = B \cup B'$ with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

   (a) Explain how achieving this would lead to a proof that $X \sim Y$.

   (b) Set $A_1 = X \setminus g(Y) = \{x \in X \,:\, x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n \,:\, n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) \,:\, n \in \mathbb{N}\}$ is a similar collection in Y.

   (c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

   (d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show that g maps B' onto A'.